ON MODULES AND RINGS WITH OPERATORS

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1) **Summary and references** — The questions contained in this paper are of some different kinds. In § 2 (modules with operators), we detail some properties of the absolute of a module, which carry on to theorem 4, where may be found a general definition of semi-linear transformation. In § 3, the writer only applies the former ideas to irreducible rings. Except theorem 5, we have no other own proposition. It follows § 4, which concerns rings with operators. One of the fundamental ideas exposed there is the notion of maximal operator domain; the definitions of admissible ideals of a ring are given in a some what different way. From the theorems of § 5, the second part of theorem 14 gives the inversion, in very general conditions, of a well known theorem. Lemma 1, already given by the writer in [23], enables us to give a common proof of a property (theorem 15) common to many radicals. There are also two propositions on the theory of representations. In § 6 (on simple rings), we may note theorem 19, which shows that a ring $\mathfrak{F}$, non zero-ring and with an operator domain, is simple if, and only if, it is simple without operators. § 7 deals with some questions on non associative algebra. We can maintain the same notion of maximal operator domain. The center of

a *naring* (non-associative ring) may be defined as the set of those elements which are operators and commute with all the elements of the ring. Theorem 19 of § 6 is then extended to simple narings (theorem 26). The purpose of § 8 is the theory of discrete direct sums. It prepares the theory of semisimple modules. We study some propositions of NAKAYAMA-AZUMAYA, [9], simplifying and continuing our former paper [24]. In § 9 we continue the theory of semisimple modules, already contained in author's paper [30]. We simplify some proofs of this work and we give very general propositions, like theorem 33, lemma 2, theorems 35 and 39, which are very useful when applied under more restritive conditions, [4], [34]. At last we give some important corrections to [30], related with theorem 40. We may note that, with respect to the simplification, we could only indicate that the proofs of pgs. 239-240 of [32] are entirely carried over the more general semisimple modules. To LÖWIG's proof (pg. 241 of the same book), contained in [33], we can give the same extension. We did so, to compare the propositions and proofs of [30] with those ones. On modules with respect to semisimple noetherian rings, we give some theorems, besides lemma 2 and theorem 35. The former are true extensions of well known WEDDERBURN-ARTIN theorems. In § 10, we return to irreducible rings. After theorems 41 and 42, we consider modules with respect to division rings $\mathfrak{D}$, which are a case of semisimple modules. We prove the simple property, that $\overline{\mathfrak{D}}$ and $\mathfrak{D}$ are reciprocal commutators in the absolute of the module, when $1 \in \mathfrak{D}$ is the identity endomorphism, not only in a direct way, but also with use of corollary 10 of theorem 31. The methods are essentialy those of [3] and [29]. We give for theorem 45, of N. JACOBSON, a formulation closely analogous to other of ARTIN-WHAPPLES, [29, pgs. 93]. Theorems 46 and 47 are obtained with proofs which, at last, are indebted to C. CHEVALLEY. The proofs already in [24] have led to an important theorem of

N. JACOBSON, then called CHEVALLEY-JACOBSON's theorem. § 11, the last one, concerns closed rings, for which we analyse two possible definitions. Theorem 50 may be considered a generalization of this one: if $\mathfrak{M} = v\mathfrak{D}$ is a module over $\mathfrak{D}$, where $1 \in \mathfrak{D}$ is the unitary operator of $\mathfrak{M}$, then $\mathfrak{D}$ and $\overline{\mathfrak{D}}$ are reciprocal commutators in the absolute of $\mathfrak{M}$. From this we can deduce a theory analogous to the one of § 10 (modules over division rings), for which we can give theorem 51: if $\mathfrak{S}$ is a noetherian simple ring of endomorphisms of a module, to which belongs the identity endomorphism, then $\mathfrak{S}$ and $\overline{\mathfrak{S}}$ are reciprocal commutators in the absolute of the module. The proof of this theorem can be given directly or by the use of corollary 10 of theorem 31. At last we give a theorem on a very special class of simple rings.

As to the references, we give only the ones not contained in [30]: [30] - A. ALMEIDA COSTA, *Über die unterdirekten Modulnsuumen,* this Revue, vol. II, 1952, pgs. 115-160; [29] - E. ARTIN and G. WHAPLES, *The theory of simple rings,* «American Journal of Mathematics», vol. LXV, 1943, pgs. 87-107; [32] - N. JACOBSON, *Lectures in Abstract Algebra,* vol. II, 1953; [33] - H. LÖWIG, *Über die Dimension linearer Räume,* «Studia Mathematica», vol. 5, 1934, pgs. 18-23; [34] - A. ALMEIDA COSTA, *Somas sub-directas de módulos, módulos semi-simples, sub-módulos-G,* «Anais da Faculdade de Ciências do Porto», vol. XXXVII, 1953; [35] - A. A. ALBERT, *On Jordan algebras of linear transformations,* «Transactions of the American Mathematical Society», vol. 59, 1946.

2) **Modules with operators** — Let $\mathfrak{M} = \{x, y, z, \cdots\}$ be a module with $\Omega = \{\lambda, \mu, \nu, \omega, \rho, \sigma, \tau, \cdots\}$ as operator domain. Then, for each $x \in \mathfrak{M}$ and $\lambda \in \Omega$, there exists a single-valued function $x\lambda$ such that: 1) $x\lambda \in \mathfrak{M}$; 2) $(x+y)\lambda = x\lambda + y\lambda$. Each operator induces an endomorphism of $\mathfrak{M}$.

We will denote by E_λ the image of λ in the absolute $\mathfrak{A}$ of the endomorphisms; by Ω_0 the image set of Ω, and by $\mathfrak{E}(\Omega_0)$ or Ω_r the subring generated by Ω_0 in $\mathfrak{A}$.

As it is easily seen, the notions of Ω-submodule and Ω_0-submodule are equivalent. Also the Ω_0-submodules are Ω_r-submodules. In fact, if $\mathfrak{m}$ is an Ω-submodule, for each $x \in \mathfrak{m}$, we have

$$(((x\lambda)\mu)\cdots)\omega = (((xE_\lambda)E_\mu)\cdots)E_\omega = xE_\lambda E_\mu \cdots E_\omega \in \mathfrak{m},$$
$$x(\Sigma \pm E_\rho E_\sigma \cdots E_\tau) = \Sigma \pm x(E_\rho E_\sigma \cdots E_\tau) \in \mathfrak{m},$$

where $\Sigma \pm E_\rho E_\sigma \cdots E_\tau$, with a finite number of summands, is the general element of the subring Ω_r, and the last sum belongs to $\mathfrak{m}$ as it happens to each summand.

The Ω-endomorphisms of $\mathfrak{M}$ are also Ω_0—or Ω_r—endomorphisms and inversely. That is: Ω_0 and Ω_r have the same commutator in $\mathfrak{A}$. If the commutator of the set $\mathfrak{C}$ of endomorphisms be denoted by $\overline{\mathfrak{C}}$, we have the following

THEOREM 1: *If $\mathfrak{M}$ is a Ω-module it is also a Ω_r-module, as it has the same set of submodules. The Ω-endomorphisms of $\mathfrak{M}$ are the Ω_r-endomorphisms and by that the commutator $\overline{\Omega}_r = \overline{\Omega}_0$ is the set of Ω-endomorphisms.*

In the following we will note, generally, by $A, B, C, \cdots, S, \cdots, X, \cdots$, the elements of $\mathfrak{A}$. Each change in notation will be carefully noted.

Consider any isomorphism $\mathfrak{M} \simeq \mathfrak{M}'$ between two modules, in which $x \in \mathfrak{M}$ and $x' \in \mathfrak{M}'$ are corresponding elements (in symbols: $x \to x'$). To some A, endomorphism of $\mathfrak{M}$, for which $x \to xA$, we may correlate an A', endomorphism of $\mathfrak{M}'$, such that $x' \to x'A' = (xA)'$. Then $\mathfrak{A}$ and $\mathfrak{A}'$, the absolutes of $\mathfrak{M}$ and $\mathfrak{M}'$, respectively, are isomorphic rings.

Let us suppose now that the modules are Ω-isomorphic, that is, they admit the same operator domain Ω, whose images in $\mathfrak{A}$ and $\mathfrak{A}'$ are Ω_0 and Ω_0', respectively. This Ω-isomorphism implies $x \to x\omega = xE_\omega$, $x' \to (x\omega)' = x'\omega = x'E_\omega'$, and the image of E_ω, in the isomorphism $\mathfrak{A} \simeq \mathfrak{A}'$, is the endomorphism E_ω'. We have therefore $x \to x'$, $xE_\omega \to x'E_\omega'$. We will say that the Ω-isomorphism denoted by $\mathfrak{M} \simeq \mathfrak{M}'$ is admissible with respect to the image sets Ω_0 and Ω_0'. It is also admissible with respect to Ω_r and Ω_r', and, generally, with respect to any two corresponding systems of endomorphisms in the isomorphism $\mathfrak{A} \simeq \mathfrak{A}'$. This isomorphism continues each one of the following ring isomorphisms: $\Omega_r \simeq \Omega_r'$; $\overline{\Omega}_r \simeq \overline{\Omega}_r'$; $\overline{\overline{\Omega}}_r \simeq \overline{\overline{\Omega}}_r'$. The last isomorphism is, in its turn, a continuation of $\Omega_r \simeq \Omega_r'$. This proves:

THEOREM 2: *If $\mathfrak{M}$ and $\mathfrak{M}'$ are isomorphic modules, this isomorphism is admissible with respect to any two systems of corresponding endomorphisms in the isomorphic absolutes: $\mathfrak{A} \simeq \mathfrak{A}'$. Any Ω-isomorphism of the modules is an admissible isomorphism, with respect to Ω_r, Ω_r', subrings generated by Ω_0, Ω_0' in the absolutes $\mathfrak{A}, \mathfrak{A}'$, respectively.*

As a special case, let us think S as an automorphism of $\mathfrak{M}$. We have $\mathfrak{M} \simeq \mathfrak{M}' = \mathfrak{M}S = \mathfrak{M}$. As we have already seen, it defines the following correspondences: $x \to x'$, $xA \to x'A'$. But now we have $x' = xS$, and, consequently, $xA \to (xA)S = x'A = (xS)A'$, that is $AS = SA'$, or $A' = S^{-1}AS$. The automorphism is an admissible one with respect to the endomorphisms A and $A' = S^{-1}AS$, which are corresponding elements in the inner automorphism $\mathfrak{A} \to S^{-1}\mathfrak{A}S = \mathfrak{A}' = \mathfrak{A}$ of the absolute $\mathfrak{A}$.

If S is an Ω-automorphism, in the same way as in theorem 2, we deduce that the correspondences $x \to x'$, $xA \to x'A'$ include $xE_\omega \to x'E_\omega'$, where $E_\omega' = S^{-1}E_\omega S$, as for A'. But as $xE_\omega = x\omega \to x'\omega = x'E_\omega = x'E_\omega'$, by hypothesis, we have

$E'_\omega = E_\omega$. The inner automorphism, which S defines, lets invariant the elements of Ω_0 and those of Ω_r and transforms the commutator $\overline{\Omega}_r$ in $\overline{\Omega}'_r = S^{-1}\overline{\Omega}_r S = \overline{\Omega}_r$, that is left globally invariant, though the same does not happen to each element. More precisely: as we have $E_\lambda = S^{-1}E_\lambda S = SE_\lambda S^{-1}$, it follows that, if $T \in \overline{\Omega}_r$, it is also $S^{-1}TS \in \overline{\Omega}_r$ and $STS^{-1} \in \overline{\Omega}_r$. Given $T \in \overline{\Omega}_r$, there exist $X \in \overline{\Omega}_r$ and $Y \in \overline{\Omega}_r$ such that $S^{-1}XS = T$ and $SYS^{-1} = T$; we may take $X = STS^{-1}$ and $Y = S^{-1}TS$. Thus we can say that $\overline{\Omega}_r \supseteq \Omega_r$ is left globally invariant in the automorphism S. Let be $Z \in \overline{\overline{\Omega}}_r$; for each $T \in \overline{\Omega}_r$, we have $ZT = TZ$. For $T = S^{-1}XS$, $(X \in \overline{\Omega}_r)$, we have $S^{-1}ZS \cdot T = S^{-1}ZS \cdot S^{-1}XS = S^{-1}ZXS = S^{-1}XZS = S^{-1}XS \cdot S^{-1}ZS = T \cdot S^{-1}ZS$ and then $S^{-1}ZS \in \overline{\overline{\Omega}}_r$. In the same way $SZS^{-1} \in \overline{\overline{\Omega}}_r$. At last, whatever may be $V \in \overline{\overline{\Omega}}_r$, there exists $W \in \overline{\overline{\Omega}}_r$ such that $S^{-1}WS = V$; we may take $W = SVS^{-1}$. Then we have the following

Theorem 3: *Let $\mathfrak{M}$ be a module and S an automorphism of $\mathfrak{M}$. Then the mapping S of $\mathfrak{M}$ onto itself is always admissible with respect to A and $S^{-1}AS$, corresponding elements in the inner automorphism of $\mathfrak{A}$ which is defined by S. If S is an Ω-automorphism, the elements of the image set Ω_0, as those of the subring Ω_r, are left invariant by S. The latter is always an $\mathfrak{S}$-automorphism, if $\mathfrak{S}$, subring of $\mathfrak{A}$, is the commutator of S.*

Sometimes we can express the first part of the theorem by saying: *every automorphism S of a module is a semi-linear transformation with respect to the absolute $\mathfrak{A}$.* This semi-linear transformation is an usual linear one, with respect to $\mathfrak{S}$. We have:

Theorem 4: *If S is an automorphism of a module, S is a semi-linear transformation with respect to every subset of the absolute which is left globally invariant by the inner automorphism defined by S or, at last, which contains, the images of its elements by the automorphism; S is a linear transformation with respect to every subset of the absolute, whose elements are left invariant by S (that is: to every subset contained in the commutator of S).*

Corollary 1: *If S is an Ω-automorphism of the module $\mathfrak{M}$, S is a semi-linear transformation with respect to $\overline{\Omega}_r$ and $\overline{\overline{\Omega}}_r$ and a linear transformation with respect to Ω_r.*

3) **Application to the irreducible ideal rings** — The module $\mathfrak{M} = (0)$ is always irreducible, whatever may be the operator ring $\mathfrak{R} = \Omega$. $\mathfrak{M} \neq (0)$ is $\mathfrak{R}$-irreducible if its $\mathfrak{R}$-submodules are only (0) and $\mathfrak{M}$. We will suppose that the product $\lambda\mu \in \mathfrak{R}$ acts according to the rule $x(\lambda\mu) = (x\lambda)\mu$. As the image set of $\mathfrak{R}$, in $\mathfrak{A}$, is $\mathfrak{R}_0 = \mathfrak{R}_r$, we can say that $\mathfrak{M}$ is $\mathfrak{R}_r$-irreducible if it is $\mathfrak{R}$-irreducible. We say that $\mathfrak{R}$ is closed if $\overline{\overline{\mathfrak{R}}}_r = \mathfrak{R}_r$, that is, if $\mathfrak{R}_r$ e $\overline{\mathfrak{R}}_r$ are reciprocal commutators in $\mathfrak{A}$, [9].

An *irreducible* ring $\mathfrak{R}$ is defined as an endomorphism ring of a module $\mathfrak{M}$, which satisfies the two conditions: 1) $\mathfrak{M}$ is faithful; 2) $\mathfrak{M}$ is $\mathfrak{R}$-irreducible. Generally, if $\mathfrak{M}$ is $\mathfrak{R}$-irreducible, $\mathfrak{R}$ is not irreducible, but $\mathfrak{R}_r$ is irreducible. $\mathfrak{R} = (0)$ is always irreducible. When $\mathfrak{R} \neq (0)$ is irreducible, we have always $\mathfrak{M}\mathfrak{R} = \mathfrak{M}\mathfrak{R}_r \neq (0)$, if $\mathfrak{R}$ is concretized by $\mathfrak{R}_r$ in the absolute $\mathfrak{A}$ of $\mathfrak{M}$. More clearly, for each $0 \neq x \in \mathfrak{M}$ we have $x\mathfrak{R} = \mathfrak{M}$ because the set of the elements $x \in \mathfrak{M}$ annihilated by $\mathfrak{R}$ is an $\mathfrak{R}$-submodule distinct from $\mathfrak{M}$. More generally, if $\mathfrak{r}$ is a right ideal of $\mathfrak{R}$, $x\mathfrak{r} \neq (0)$ implies $x\mathfrak{r} = \mathfrak{M}$.

Let us suppose that the irreducible ring $\mathfrak{R}$ has a minimal right ideal $\mathfrak{r}$. If $\mathfrak{M}$ is a faithful module, we have $\mathfrak{M}\mathfrak{r} \neq (0)$. If $x_0 \in \mathfrak{M}$ is such that $x_0\mathfrak{r} \neq (0)$, we may consider the correspondence $r \to x_0 r$, where $r \in \mathfrak{r}$. It is an $\mathfrak{R}$-isomorphism and, as $x_0\mathfrak{r}=\mathfrak{M}$, it follows that $\mathfrak{M}$ is $\mathfrak{R}$-isomorphic to every minimal right ideal of $\mathfrak{R}$. Then the minimal right ideals are isomorphic to each other and isomorphic to each module in whose absolute we can concretize $\mathfrak{R}$, [9], [26].

Let us take $\mathfrak{M}\simeq\mathfrak{r}$. If $\mathfrak{A}$ and $\mathfrak{A}'$ are their absolutes, the isomorphism $\mathfrak{A}\simeq\mathfrak{A}'$ continues the isomorphism $\mathfrak{R}\simeq\mathfrak{R}'$, where $\mathfrak{R}'$ is the image subring of $\mathfrak{R}$ in $\mathfrak{A}'$. Then, we may suppose that $\mathfrak{r}$ is an $\mathfrak{R}$-module, which concretizes the irreducible ring $\mathfrak{R}$. If, in particular, it is $\mathfrak{M}=\mathfrak{r}$, let us take $x_0 \in \mathfrak{r}$ such that $x_0\mathfrak{r}\neq(0)$. The isomorphism $r\rightarrow x_0 r, (r \in \mathfrak{r})$, already considered, shows that there is some $r' \in \mathfrak{r}$ such that $x_0 r'=x_0$, $x_0 r'^2=x_0$. The images of r' and r'^2 are the same and then $r'=r'^2$. The element $r'=e$ is a non-null idempotent of $\mathfrak{r}$. The irreducible rings with minimal right ideals are called, in [9], *irreducible ideal rings*.

As every idempotent of a minimal regular ideal is a primitive one, [(I), pgs. 18-19], every minimal right ideal of an irreducible ideal ring can be generated by a primitive idempotent. For such a ring $\mathfrak{R}$, let $\mathfrak{R}_r$ and $\mathfrak{R}'_r$ be two concretizations of $\mathfrak{R}$ in the absolutes $\mathfrak{A}$ and $\mathfrak{A}'$ of two modules $\mathfrak{M}$ and $\mathfrak{M}'$. As $\mathfrak{M}$ and $\mathfrak{M}'$ are $\mathfrak{R}$-isomorphic, we know, by theorem 2, that to the isomorphism $\mathfrak{M}\simeq\mathfrak{M}'$ we can give the following meaning, [5], [26]: $x\longleftrightarrow x'$, $xR\longleftrightarrow x'R'$, $xD\longleftrightarrow x'D'$, if $R \in \mathfrak{R}_r$ and $R' \in \mathfrak{R}'_r$ are the images of the same element $\rho \in \mathfrak{R}$, and $D \in \overline{\mathfrak{R}}_r=\mathfrak{D}$ and $D' \in \overline{\mathfrak{R}}'_r=\mathfrak{D}'$ are corresponding elements in the isomorphism $\mathfrak{A}\simeq\mathfrak{A}'$. Clearly, $\mathfrak{D}$ and $\mathfrak{D}'$ are division rings.

As we can use the theorems 3 and 4, we vill prove the following

Theorem 5: *Let $\mathfrak{R}$ be an irreducible ideal ring with two concretizations $\mathfrak{R}_r$ and $\mathfrak{R}'_r$, in the absolute $\mathfrak{A}$ of some module $\mathfrak{M}$. There are elements $\rho_0 \in \mathfrak{r}$ (a minimal right ideal of $\mathfrak{R}$), whose correspondents $r \in \mathfrak{r}_r$ and $r' \in \mathfrak{r}'_r$ (where $\mathfrak{r}_r$ and $\mathfrak{r}'_r$ are the images of $\mathfrak{r}$) are different elements.* Given $\rho \in \mathfrak{R}$, we will take $x\rho=xR$ and $x\rho=xR'$ for each concretization.

Let us suppose that we have always $r=r'$ and that $0\neq x_0 \in \mathfrak{M}$ satisfies $x_0\mathfrak{r}=x_0\mathfrak{r}_r=x_0\mathfrak{r}'_r=\mathfrak{M}$. For every $x \in \mathfrak{M}$, we have $x=x_0\rho_0=x_0 r=x_0 r'$, $(r'=r)$. Then, it is $x\rho=(x_0\rho_0)\rho=$ $=x_0(\rho_0\rho)$. As $\rho_0\rho \in \mathfrak{r}$, it follows $x\rho=x_0(\rho_0\rho)=x_0(rR)=$ $=x_0(r'R')$, and thus $xR=xR'$, for every x and ρ, which is absurd.

Let us suppose $\mathfrak{r}\simeq\mathfrak{r}_r\simeq x_0\mathfrak{r}_r=\mathfrak{M}$, $\mathfrak{r}\simeq\mathfrak{r}'_r\simeq x'_0\mathfrak{r}'_r=\mathfrak{M}$, and consider S, automorphism of $\mathfrak{M}$, by which $x_0 r\rightarrow x'_0 r'$, $x_0 rR\rightarrow x'_0 r'R'$, as we see in the scheme

$$\rho_0 \begin{array}{l} \nearrow r \rightarrow x_0 r \\ \qquad\quad \updownarrow \\ \searrow r' \rightarrow x'_0 r' \end{array}, \qquad \rho_0\rho \begin{array}{l} \nearrow rR \rightarrow x_0 rR \\ \qquad\qquad \updownarrow \\ \searrow r'R' \rightarrow x'_0 r'R' \end{array}.$$

As we have $(x_0 r)S=x'_0 r'$, $(x_0 rR)S=x'_0 r'R'=(x_0 r\cdot S)R'$, we conclude $RS=SR'$, or $R'=S^{-1}RS$. Then S gives a correspondence between the two image rings of $\mathfrak{R}$ in $\mathfrak{A}$. The following proposition holds: Let $\mathfrak{M}$ be a module whose absolute contains two concretizations, $\mathfrak{R}_r$ and $\mathfrak{R}'_r$, of an irreducible ideal ring $\mathfrak{R}$. If the isomorphism $\mathfrak{R}_r\simeq\mathfrak{R}'_r$ determines the correspondence $R\longleftrightarrow R'$, there exists S, automorphism of $\mathfrak{M}$, by which $x\rightarrow xS=x'$, $R'=S^{-1}RS$, that is, $x\rightarrow x'$, $xR\rightarrow(xR)S=x'R'$.

In the terminology of theorem 3, we can give the following addendum: The automorphism S is an admissible one with respect to the commutators $\mathfrak{D}$ and $\mathfrak{D}'$, which are in correspondence in the absolute, such that, if $D\rightarrow D'=S^{-1}DS$, also $xD\rightarrow xDS=xSD'=x'D'$.

Jacobson, [5], gives a proposition for the hypothesis $\mathfrak{D}'=\mathfrak{D}$. By the theorem 4, we conclude that S is a semi-linear transformation with respect to $\mathfrak{D}$.

4) **Rings with operators** — Let $\mathfrak{S}=\{a,b,c,d,\cdots,r,s,t,v,\cdots\}$ be a ring with an operator domain $\Omega=\{\lambda,\mu,\nu,\omega,\rho,\sigma,\tau,\cdots\}$. Besides properties 1) $a\omega \in \mathfrak{S}$; 2) $(a+b)\omega=a\omega+b\omega$; already

accepted for the module $\mathfrak{S}$, [§ 2], we have also: 3) $(ab)\omega = (a\omega)b = a(b\omega)$. By $\mathfrak{A}$ we also mean the absolute of $\mathfrak{S}$ and by $\overline{\mathfrak{C}}$ the commutator of some set $\mathfrak{C}$ of endomorphisms contained in $\mathfrak{A}$. Also Ω_0 and Ω_r will have the same meaning of § 2.

We will denote by $E_s^{(r)}, E_t^{(r)}, \cdots$ the images, in $\mathfrak{A}$, of the endomorphisms induced by $s, t, \cdots \in \mathfrak{S}$, when they are used as right multipliers in $\mathfrak{S}$, and by $E_s^{(l)}, E_t^{(l)}, \cdots$ the images, in $\mathfrak{A}$, of the endomorphisms induced by $s, t, \cdots \in \mathfrak{S}$, when used as left multipliers in $\mathfrak{S}$. The set of the $E_s^{(r)}$, $(s \in \mathfrak{S})$, is a ring $\mathfrak{S}_r$, homomorphic image of $\mathfrak{S}$, and the set of the $E_s^{(l)}$, $(s \in \mathfrak{S})$, is also a ring $\mathfrak{S}_l$, anti-homomorphic of $\mathfrak{S}$.

From $(ab)\lambda = (aE_b^{(r)})E_\lambda = (a\lambda)b = (aE_\lambda)E_b^{(r)}$, $(ab)\lambda = (bE_a^{(l)})E_\lambda = a(b\lambda) = (bE_\lambda)E_a^{(l)}$, we conclude that $\Omega_r \subseteq \overline{\mathfrak{S}}_r$, $\Omega_r \subseteq \overline{\mathfrak{S}}_l$, and, by that, $\Omega_r \subseteq \overline{\mathfrak{S}}_r \cap \overline{\mathfrak{S}}_l$. On the other hand, $sa \cdot t = (aE_s^{(l)})E_t^{(r)} = s \cdot at = (aE_t^{(r)})E_s^{(l)}$ shows that $\mathfrak{S}_r \subseteq \overline{\mathfrak{S}}_l$, $\mathfrak{S}_l \subseteq \overline{\mathfrak{S}}_r$. The product $\mathfrak{S}_r\mathfrak{S}_l = \mathfrak{S}_l\mathfrak{S}_r$, does not contain, generally, the factors. Let $\mathfrak{T} = \mathfrak{G}(\mathfrak{S}_r, \mathfrak{S}_l)$ be the subring of $\mathfrak{A}$ generated by $\mathfrak{S}_r$ and $\mathfrak{S}_l$. We have $\overline{\mathfrak{T}} = \overline{\mathfrak{S}}_r \cap \overline{\mathfrak{S}}_l$ and consequently $\Omega_r \subseteq \overline{\mathfrak{T}}$. We may note that $\Omega_r \subseteq \overline{\mathfrak{T}}$ is a consequence of 3). Therefore the ring $\overline{\mathfrak{T}}$ acts as a maximal operator domain of $\mathfrak{S}$. Those elements $s \in \mathfrak{S}$, which may act as operators, are characterized by the property $E_s^{(r)} \in \overline{\mathfrak{T}}$ and they form a subring $\mathfrak{B}$, of $\mathfrak{S}$. We have then:

Theorem 6: *For the ring $\mathfrak{S}$, with operator domain Ω, we can verify the followiug properties in the absolute of its module: $\Omega_r \subseteq \overline{\mathfrak{S}}_r \cap \overline{\mathfrak{S}}_l = \overline{\mathfrak{T}}$, $[\mathfrak{S}_r \subseteq \overline{\mathfrak{S}}_l, \mathfrak{S}_l \subseteq \overline{\mathfrak{S}}_r]$, and $\overline{\mathfrak{T}}$ is the maximal operator domain of $\mathfrak{S}$. $s \in \mathfrak{S}$ satisfies $E_s^{(r)} \in \overline{\mathfrak{T}}$, if, and only if, $s \in \mathfrak{B} (\subseteq \mathfrak{S})$, subring whose elements t satisfy the relations $(xy)t = (xt)y = x(yt)$, with arbitrary $x, y \in \mathfrak{S}$.*

Corollary 2: *If $\mathfrak{S}$ is a zero-ring ($\mathfrak{S}^2 = (0)$), its maximal operator domain is the absolute. We have $\mathfrak{A} = \overline{\mathfrak{T}} = \overline{\mathfrak{S}}_r = \overline{\mathfrak{S}}_l$.*

This is an immediate consequence of the following equalities: $\mathfrak{S}_r = (0) = \mathfrak{S}_l$.

Obviously, the center $\mathfrak{Z}$, of $\mathfrak{S}$, is contained in the subring $\mathfrak{B}$, and $\mathfrak{Z}_r \subseteq \overline{\mathfrak{T}}$. Also, we may note that we have $\mathfrak{T}\overline{\mathfrak{T}} = \overline{\mathfrak{T}}\mathfrak{T} \subseteq \mathfrak{T}$, $\mathfrak{S}_r\Omega_r \subseteq \mathfrak{S}_r$, $\mathfrak{S}_l\Omega_r \subseteq \mathfrak{S}_l$, and, consequently, $\mathfrak{S}_r\overline{\mathfrak{T}} \subseteq \mathfrak{S}_r$, $\mathfrak{S}_l\overline{\mathfrak{T}} \subseteq \mathfrak{S}_l$.

Let $s, s' \in \mathfrak{B}$ be given and let us consider the commutator $[ss'] = ss' - s's$. For every $a \in \mathfrak{S}$, we have $a[ss'] = 0$. If there exists some a which is not a zero left-divisor, we conclude $[ss'] = 0$, that is, the commutativity of $\mathfrak{B}$. Generally, $a[ss'] = 0$, only implies the commutativity of $\mathfrak{B}_r$.

Let us consider $\mathfrak{S}^2 = \mathfrak{S}$. For every $x \in \mathfrak{S}$, we have $x = \Sigma aa'$, with $a, a' \in \mathfrak{S}$. For $T, T' \in \overline{\mathfrak{T}}$, we have $(aa')(TT') = (aT)(a'T') = (a \cdot a'T)T' = (aT')(a'T) = (aa')(T'T)$, and, consequently, $x(TT') = x(T'T)$. $\overline{\mathfrak{T}}$ is then a commutative ring and the same is true for every operator domain Ω_r. Then:

Theorem 7: *If $\mathfrak{S}$ is such that $\mathfrak{S}^2 = \mathfrak{S}$, its operator domain acts in a commutative way; and if $\mathfrak{S}$ has not left zero-divisors, the subring $\mathfrak{B}$ is a commutative one.*

Let us suppose now that there exists an identity $u \in \mathfrak{S}$. The image of u, in $\mathfrak{A}$, is the identity endomorphism. It is easily seen that $\mathfrak{S}_r$ and $\mathfrak{S}_l$ are reciprocal commutators in $\mathfrak{A}$. We know that $\mathfrak{S}_l \subseteq \overline{\mathfrak{S}}_r$. If $\sigma \in \mathfrak{A}$ is such that $(ba)\sigma = (b\sigma)a$, for $b = u$ we have $a\sigma = (u\sigma)a = ca = aE_c^{(l)}$, where $u\sigma = c \in \mathfrak{S}$. Then $\sigma \in \mathfrak{S}_l$, and consequently $\overline{\mathfrak{S}}_r \subseteq \mathfrak{S}_l$, $\mathfrak{S}_l = \overline{\mathfrak{S}}_r$. Then $\overline{\overline{\mathfrak{S}}}_r = \overline{\mathfrak{S}}_l = \mathfrak{S}_r$ and we have $\Omega_r \subseteq \overline{\mathfrak{T}} = \mathfrak{S}_l \cap \mathfrak{S}_r$. The application of every operator is equivalent to the application of the same element of the ring in each side. In fact, if we put $a = u$ in $aT = ba = ac$, we have $b = c$, and, for every $a \in \mathfrak{S}$, $ab = ba$, which shows that $b \in \mathfrak{Z}$. This proves:

Theorem 8: *In a ring with identity, the center is the maximal operator domain, and has in the absolute the image* $\mathfrak{S}_r \cap \mathfrak{S}_l$. *Such ring is closed.*

The last hypothesis about Ω is the following one. Let $\mathfrak{S}$ be a ring, Ω_r its operator domain and $\overline{\mathfrak{T}}$ the maximal operator domain. Let us suppose that $\varepsilon \in \overline{\mathfrak{T}}$ is the identity of $\overline{\mathfrak{T}}$. For every $s \in \mathfrak{S}$, we have $s = s\varepsilon + s(1-\varepsilon)$, where 1 is the identity endomorphism. The set of the elements $s\varepsilon$ is an Ω_r-ideal, as we see from the equalities: $a \cdot s\varepsilon = (as)\varepsilon$, $s\varepsilon \cdot a = (sa)\varepsilon$, $s\varepsilon \cdot E_\lambda = s \cdot \varepsilon E_\lambda = sE_\lambda \cdot \varepsilon = (s\lambda)\varepsilon$, because $\varepsilon \in \overline{\mathfrak{T}}$. We can say the same about the set of elements $s - s\varepsilon$. Then $\mathfrak{S} = \mathfrak{S}\varepsilon + \mathfrak{S}(1-\varepsilon)$ is a direct sum of two ideals.

Theorem 9: *It is a necessary and sufficient condition to be* $\varepsilon = 1$, *that* $\overline{\mathfrak{T}}A = A\overline{\mathfrak{T}} \neq (0)$, *for every* $0 \neq A \in \overline{\mathfrak{T}}$. If $\varepsilon = 1$, then $A \in \overline{\mathfrak{T}}A$ and $\overline{\mathfrak{T}}A \neq (0)$. Conversely, if, for every $A \neq 0$, is $\overline{\mathfrak{T}}A \neq (0)$, as $\overline{\mathfrak{T}}(1-\varepsilon) = \overline{\mathfrak{T}}\varepsilon(1-\varepsilon) = (0)$ and $1-\varepsilon \in \overline{\mathfrak{T}}$, we have $1-\varepsilon = 0$ or $\varepsilon = 1$. We can give another formulation of this theorem.

Theorem 10: *It is a necessary and sufficient condition for* $\varepsilon = 1$, *that* $\mathfrak{S}$ *does not have absolute zero-divisors (that is, elements* $a \neq 0$ *such that* $ax = xa = 0$ *for every* $x \in \mathfrak{S}$), [35]. If $\varepsilon = 1$, $a \neq 0$ is not an absolute zero divisor, because we would have $a\overline{\mathfrak{T}} = (0)$, $a\varepsilon = a = 0$. Conversely, let be $0 \neq A \in \overline{\mathfrak{T}}$. There exists $x \in \mathfrak{S}$ such that $xA \neq 0$. As xA is not an absolute zero divisor, we have $(xA)\overline{\mathfrak{T}} = (x\overline{\mathfrak{T}})A \neq (0)$. Then $\overline{\mathfrak{T}}A \neq (0)$ and $\varepsilon = 1$, by theorem 9.

By the former considerations, we can understand easily the meaning of the admissible right ideal, generated by a set of elements of $\mathfrak{S}$, whatever may be the operator domain Ω. We have:

Theorem 11: *The right ideal* $(a)_r$, *generated by* a, *is the set of elements of* $\mathfrak{S}$ *of the form* $a(\Sigma \pm 1 E_\lambda \cdots E_\mu E_s^{(r)} \cdots E_t^{(r)}) = a(\Sigma \pm 1 E_s^{(r)} \cdots E_t^{(r)} E_\lambda \cdots E_\mu) = a\mathfrak{E}(1, \Omega_r, \mathfrak{S}_r)$. In this notation, 1 is the identity endomorphism of $\mathfrak{S}$ and Σ has only a finite number of summands. In particular, we have, in $(a)_r$, $a(1) = a$, $a(-1) = -a$, $a(1 + \cdots + 1) = ma$, $aE_s^{(r)} = as$, $aE_\lambda = a\lambda$, $a(E_\lambda E_s^{(r)}) = (a\lambda)s = (as)\lambda$ and also $(((a\lambda)\mu)\cdots)\omega = a(E_\lambda E_\mu \cdots E_\omega)$. More generally, we can say:

Theorem 12: *The right ideal of* $\mathfrak{S}$ *generated by* $\mathfrak{C} = \{a, b, c, \cdots, a', b' c', \cdots\} \subseteq \mathfrak{S}$ *is the set of elements of* $\mathfrak{S}$ *of the form* $a(\Sigma \pm 1 E_\lambda \cdots E_\omega E_s^{(r)} \cdots E_t^{(r)}) + \cdots + a'(\Sigma \pm 1 E_\mu \cdots E_\tau E_v^{(r)} \cdots E_w^{(r)})$, *where there is a finite number of* $\Sigma\Sigma$, *each one with a finite number of summands.*

With respect to two-sided ideals, we can give the following

Theorem 13: *The ideal* (a), *generated by* a, *is the set of elements with the form* $a\mathfrak{E}(1, \Omega_r, \mathfrak{S}_r, \mathfrak{S}_l) = a\mathfrak{E}(1, \Omega_r, \overline{\mathfrak{T}})$.

We may note the following: from $s \cdot a\lambda = sE_{a\lambda}^{(r)} = (sa)\lambda = sE_a^{(r)} E_\lambda$, we conclude $E_{a\lambda}^{(r)} = E_a^{(r)} E_\lambda$, and hence we have $\pm 1 E_s^{(r)} \cdots E_t^{(r)} E_v^{(r)} E_\lambda E_\mu \cdots E_\tau = \pm 1 E_s^{(r)} \cdots E_t^{(r)} E_{v\lambda}^{(r)} E_\mu \cdots E_\tau = \pm 1 E_s^{(r)} \cdots E_t^{(r)} E_{(v\lambda)\mu}^{(r)} \cdots E_\tau$, and so on. But it seems preferable to maintain the notation used in theorems 11 and 12. A second note may be this: if $\mathfrak{r}$ is a right ideal, the ideal $\mathfrak{r}^2$ can be obtained without the use of operators. And a left ideal of the form $\mathfrak{S}a$ is always an admissible one (as $a\mathfrak{S}$) because $\mathfrak{S}a = a\mathfrak{S}_l$, $(\mathfrak{S}a)\Omega_r = (a\mathfrak{S}_l)\Omega_r \subseteq a\mathfrak{S}_l$.

We can now give in a simple way the notion of «root» of a ring, as it is used in the theory of the classical radical: a is a root in the ring with operators $\mathfrak{F}$, if $(a)_r = a\mathfrak{E}(1, \Omega_r, \mathfrak{F}_r)$ is a nilpotent right ideal.

A last note is that one: if there exists an identity, we have $(a)_r = a\mathfrak{G}(\mathfrak{F}_r) = a\mathfrak{F}_r$, $(a) = a\mathfrak{E}(\mathfrak{T}) = a\mathfrak{T} = a\mathfrak{F}_r\mathfrak{F}_l = a\mathfrak{F}_l\mathfrak{F}_r$.

5) **Some general theorems** — As we have already seen, if $\mathfrak{F}$ is an associative ring with any operator domain Ω, the left ideal generated by the idempotent e is $\mathfrak{F}e = e\mathfrak{F}_l$. The ring of $\mathfrak{F}$-endomorphisms of $\mathfrak{F}e$ is isomorphic to $e\mathfrak{F}e$. The $\mathfrak{F}$-endomorphisms are also Ω-endomorphisms. If $e\mathfrak{F}e$ is a division ring, it follows, from a general property of endomorphisms with inverse, that every $\mathfrak{F}$-endomorphism of $\mathfrak{F}e$ is an automorphism. In correlation, we have the following

Theorem 14: *If $\mathfrak{F}e$ has no nilpotent admissible left ideal of $\mathfrak{F}$, and, for each $0 \neq a \in \mathfrak{F}e$, is $\mathfrak{F}a \neq (0)$, then $\mathfrak{F}e$ is minimal, if, and only if, $e\mathfrak{F}e$ is a division ring.* If $\mathfrak{F}e$ is $(\mathfrak{F}, \Omega)$-minimal, the ring of its $(\mathfrak{F}, \Omega)$-endomorphisms, or $\mathfrak{F}$-endomorphisms, is a division ring. Conversely, if $e\mathfrak{F}e$ is a division ring, and $0 \neq a \in \mathfrak{F}e$, as $e\mathfrak{F}e \cdot e\mathfrak{F}a \subseteq e\mathfrak{F}a$, we conclude that the left ideal $e\mathfrak{F}a$ of $e\mathfrak{F}e$ satisfies one of the following equalities: $e\mathfrak{F}a = (0)$ or $e\mathfrak{F}a = e\mathfrak{F}e$. If $e\mathfrak{F}a = (0)$, it would be $\mathfrak{F}e\mathfrak{F}a = (0)$, and, consequently, $(\mathfrak{F}a)^2 = (0)$ and $\mathfrak{F}a = (0)$, which is absurd. Then we have $e\mathfrak{F}a = e\mathfrak{F}e$, $\mathfrak{F}e\mathfrak{F}a = \mathfrak{F}e\mathfrak{F}e = \mathfrak{F}e \subseteq \mathfrak{F}a$, which shows that $\mathfrak{F}e$ is minimal, $[\mathfrak{F}e = \mathfrak{F}a]$.

An useful consequence is the following: In a ring without nilpotent ideals, $\mathfrak{F}e$ is minimal if, and only if, $e\mathfrak{F}e$ is a division ring. In fact, for $a \neq 0$, we have $\mathfrak{F}a \neq (0)$, because $\mathfrak{F}a = (0)$ implies that there exists an admissible ideal $\mathfrak{a} \neq (0)$, such that $\mathfrak{F}\mathfrak{a} = (0)$.

Evidently, we have an analogue of theorem 14, and its consequence, for the minimal right ideal $e\mathfrak{F}$. An obvious consequence is that in a ring without nilpotent ideal, if $\mathfrak{F}e$ is a minimal left ideal, then $e\mathfrak{F}$ is a minimal right ideal, [8, pgs. 13].

If we call semi-simple ring, with respect to some radical, any ring whose radical is the null ideal, we can speak of semi-simple rings in the sense of Levitzki, Köthe, Jacobson and Brown-McCoy, for instance, (Cf. [I, Cap. I] and [23], [25], [26], [27]). Also the upper radical of Baer, [23, pgs. 104], denoted by $\mathfrak{U}$, is such that $\mathfrak{F}/\mathfrak{U}$ has the null ideal as the upper radical. In those semi-simple rings and in those rings with $\mathfrak{U} = (0)$, we can use Theorem 14 and its consequences. A property common to $\mathfrak{U}$ and the others radicals is the following one (in which $\mathfrak{W}$ denotes any of those radicals):

Theorem 15: *It is a necessary and sufficient condition for $a \in \mathfrak{W}$, that $a\mathfrak{F} \subseteq \mathfrak{W}$.* This is a consequence of the following

Lemma 1: *If $\mathfrak{P}$ be an admissible ideal such that $\mathfrak{F}/\mathfrak{P}$ has no admissible nilpotent ideal, then $a \in \mathfrak{P}$, if, and only if, $a\mathfrak{F} \subseteq \mathfrak{P}$.*

Evidently, this is a necessary condition. Conversely, let $\mathfrak{F}$ be a ring with the operator domain Ω and let us suppose $a\mathfrak{F} \subseteq \mathfrak{P}$. If $\mathfrak{r}$ is the right ideal $(a)_r$, generated by a, the ideal $\mathfrak{r}^2$ is the set of elements $\Sigma a'a''$, as it was already referred. Then we have $\mathfrak{r}^2 \subseteq a\mathfrak{F} \subseteq \mathfrak{P}$. If $\mathfrak{r}'$ is the image of $\mathfrak{r}$ in the homomorphism $\mathfrak{F} \sim \mathfrak{F}/\mathfrak{P}$, we have $\mathfrak{r}'^2 = (0)$, $\mathfrak{r}' = (0)$, and consequently $\mathfrak{r} \subseteq \mathfrak{P}$, $a \in \mathfrak{P}$.

In the sequence of the general considerations of this paragraph, we will study a subject connected with the representation theory, [(I), Cap. VIII, pgs. 222-223].

For a given module $\mathfrak{M}$, let $\mathfrak{S}$ and $\mathfrak{R}$ be two right operator rings such that $\mathfrak{M}$ is an $\mathfrak{S}$-module and $\mathfrak{R}$-module. The image sets of $\mathfrak{S}$ and $\mathfrak{R}$, in the absolute $\mathfrak{A}$, will be denoted (as it is our general use) by $\mathfrak{S}_r$ and $\mathfrak{R}_r$. We will say that $\mathfrak{M}$ is a double right $(\mathfrak{S}, \mathfrak{R})$-module if $\mathfrak{S}_r \subseteq \overline{\mathfrak{R}}_r$, and, consequently, $\mathfrak{R}_r \subseteq \overline{\mathfrak{S}}_r$. If $\mathfrak{R}$ has a faithful representation in $\mathfrak{A}$ $(\mathfrak{R} \simeq \mathfrak{R}_r)$, the image ring $\mathfrak{S}_r$ is said a direct representation of $\mathfrak{S}$ in $\mathfrak{R}$, and $\mathfrak{M}$ is the representation module. Generally, however, $\mathfrak{S}_r$ is a direct representation of $\mathfrak{S}$ in $\mathfrak{R}_r$. For $v \in \mathfrak{M}$, $a \in \mathfrak{S}$, $\lambda \in \mathfrak{R}$, we have the exchange law $vE_a E_\lambda = vE_\lambda E_a$, or $va \cdot \lambda = v\lambda \cdot a$.

If $\mathfrak{S}$ has an operator domain $\Omega \subseteq \mathfrak{R}$, the direct representation is said admissible, if we have also, for $\varrho \in \Omega$, $E_{a\varrho} = E_a E_\varrho$. This product makes sense, but though $E_{a\varrho}$ and E_a are elements of $\overline{\mathfrak{R}}_r$, we cannot say the same about E_ϱ. With respect to the representation module, we have $v \cdot a\varrho = va \cdot \varrho = v\varrho \cdot a$, together with the general relation $va \cdot \lambda = v\lambda \cdot a$. These relations are sufficient (with a short extension of the notion of admissible representation) to give an admissible representation of the Ω-ring $\mathfrak{S}$, $(\Omega \subseteq \mathfrak{R})$, in $\mathfrak{R}_r$. Let us suppose that $\mathfrak{M}\mathfrak{S} = \mathfrak{M}\mathfrak{S}_r = \mathfrak{M}$. For $v \in \mathfrak{M}$ we have then $v = \Sigma m_i \cdot E_{a_i}$, with $m_i \in \mathfrak{M}$, $E_{a_i} \in \mathfrak{S}_r$. It will be $vE_\varrho \lambda = \Sigma (m_i E_{a_i}) E_\varrho \lambda = \Sigma m_i \lambda E_{a_i \varrho} = \Sigma m_i \lambda E_{a_i} E_\varrho = = \Sigma m_i E_{a_i} \lambda E_\varrho = v\lambda E_\varrho$. Consequently, we have $vE_\varrho E_\lambda = = vE_\lambda E_\varrho$, that is, $E_\varrho \in \overline{\mathfrak{R}}_r$. If $\mathfrak{R}$ has a faithful representation in $\mathfrak{A}$, we have $\varrho\lambda = \lambda\varrho$, and Ω is contained in the center of $\mathfrak{R}$. We have then

THEOREM 16: *Let $\mathfrak{S}$ be an Ω-ring. If $\mathfrak{S}$ has an admissible representation in $\mathfrak{R}_r$, the hypothesis $\Omega \subseteq \mathfrak{R}$ and $\mathfrak{M}\mathfrak{S} = \mathfrak{M}$, implies $\Omega_r \subseteq \overline{\mathfrak{R}}_r \cap \mathfrak{R}_r$, which is the center of $\mathfrak{R}_r$. And, if $\mathfrak{R}$ has a concretization in $\mathfrak{A}$, Ω is contained in the center of $\mathfrak{R}$.*

Let us study now the finite representations. This signifies that the representation module is finite over $\mathfrak{R}$,

i. e., it is of the form $\mathfrak{M} = u_1 \mathfrak{R} + \cdots + u_n \mathfrak{R}$, where we suppose that $\mathfrak{R}$ has an identity, which is the unitary operator of the module, and the u_i are a $\mathfrak{R}$-independent basis of $\mathfrak{M}$. As $\mathfrak{R}$ is an anti-isomorphic ring of $\mathfrak{R}_n$ (full matrix ring over $\mathfrak{R}$), the direct representation of $\mathfrak{S}$, that we have considered in a general way, can be substituted by a reciprocal one by matrices of $\mathfrak{R}_n$. To $a \in \mathfrak{S}$ will correspond the matrix $A \in \mathfrak{R}_n$, defined by

$$u_i a = \sum_{j=1}^{n} u_j \alpha_{ji}, \quad A = (\alpha_{ji}), \quad \alpha_{ji} \in \mathfrak{R}.$$

If there exists an operator domain $\Omega \subseteq \mathfrak{R}$, and the representation is admissible, we will have the correspondence $a \to A$, $a\varrho \to A\varrho$, as it follows from

$$u_i \cdot a\varrho = u_i E_{a\varrho} = u_i E_a E_\varrho = (\sum_{j=1}^{n} u_j \alpha_{ji}) E_\varrho = \sum_{j=1}^{n} u_j (\alpha_{ji}\varrho).$$

Let us consider next the converse hypothesis. We know that the Ω-ring $\mathfrak{S}$, $(\Omega \subseteq \mathfrak{R})$, has a reciprocal representation by matrices in $\mathfrak{R}_n$, by which $a \to A$, $a\varrho \to A\varrho$. Thinking in the module $\mathfrak{M} = u_1 \mathfrak{R} + \cdots + u_n \mathfrak{R}$, we put in correspondence the matrices A and $A\varrho$ with $\mathfrak{R}$-endomorphisms, represented by E_a and $E_{a\varrho}$, respectively.

We have not necessarily $E_{a\varrho} = E_a E_\varrho$, as for $v = = \Sigma u_i \lambda_i$, we have the following equalities: $v \cdot a\varrho = = \sum_i (u_i \lambda_i) a\varrho = \sum_i (u_i \cdot a\varrho) \lambda_i = \sum_{i,j} u_j \alpha_{ji} \varrho \lambda_i$ and

$$va \cdot \varrho = \sum_i (u_i \lambda_i \cdot a) \varrho = \sum_{i,j} u_j \alpha_{ji} \lambda_i \varrho,$$

that show that, in general, $v \cdot a\varrho \neq va \cdot \varrho$. Corresponding to this result, let us consider, if possible, a change of basis in $\mathfrak{M}$, e. g., $(u'_1, \cdots, u'_n) = (u_1, \cdots, u_n) . P$, where P is an invertible matrix. The matrices $P^{-1}AP$ and $P^{-1}(A\varrho)P$ induce in the new basis the same endomorphisms E_a and $E_{a\varrho}$. Generally, however, it is $P^{-1}(A\varrho)P \neq (P^{-1}AP)\varrho$, differently from what happens in the first basis. But we have this

Theorem 17: *To every finite module over* $\mathfrak{K}$ *of an admissible representation, in* $\mathfrak{K}$, *of an* Ω-*ring* $\mathfrak{S}$, $(\Omega \subseteq \mathfrak{K})$, *corresponds an admissible representation by finite matrices, and, conversely, if* $\mathfrak{K}_r$ *is commutative, (or, at least,* Ω_r *is contained in center of* $\mathfrak{K}_r$*), the existence of the last one representation carries the existence of the first one, and, therefore, the existence of the correspondent representation module.*

6) **On simple rings** — Let us begin by the study of a simple zero-ring without operators. Then $\mathfrak{S}$ is a commutative ring because $ab = ba = 0$ for every a and b. The ideal generated by $a \neq 0$ has the form $\{ma\}$, where m is an integer. Then $\mathfrak{S} = \{ma\}$, and for every $ka \neq 0$, there is an integer r such that $a = r \cdot ka$. From $(rk-1)a = 0$, we conclude that $\mathfrak{S} = \{0, a, 2a, \cdots, (q-1)a\}$ is a finite group and the finite characteristic q is a prime number. The absolute $\mathfrak{A}$ of the endomorphisms of $\mathfrak{S}$, all $\mathfrak{S}$-endomorphisms, is the commutator of $0 \in \mathfrak{A}$, which is the image of $\mathfrak{S}$. More precisely, $\mathfrak{A}$ is the field $I/(q)$, where I is the ring of integers.

Let us suppose that $\mathfrak{S}$, also a zero-ring, is simple with respect to an operator domain Ω, with an element which does not induce the null endomorphism. Every Ω-submodule is an ideal, and $\mathfrak{S}$ is an Ω-simple module. Every element of the absolute can be considered as an operator, as we have seen in § 4. $\mathfrak{S}$ is a Ω_r-simple module. The commutator of Ω_r, or of every subring of $\mathfrak{A}$ containing Ω_r, is a division ring. Consequently, the commutator of $\mathfrak{A}$ (the center of $\mathfrak{A}$) is a field. If $a\Omega_r \neq (0)$, the structure

of $\mathfrak{S}$ is given by the relation $\mathfrak{S} = a\Omega_r$. When Ω is a commutative domain, Ω_r is a commutative ring and is contained in $\bar{\Omega}_r$. For every $0 \neq a \in \mathfrak{S}$, and $0 \neq A \in \Omega_r$, we have $aA \neq 0$. Taking $B \in \bar{\Omega}_r$ and $aB = aB_0$, with $B_0 \in \Omega_r$, we have $B = B_0$ and $\Omega_r = \bar{\Omega}_r$. We have then:

Theorem 18: *If* $\mathfrak{S}$ *is a zero ring, simple with respect to a commutative operator domain* Ω, *the commutator of* $\Omega_r \neq (0)$ *is* Ω_r. From this theorem and the result for a void set Ω, we conclude:

Corollary 3: *Every* Ω-*simple zero ring, where* Ω *is a commutative domain, is a simple zero algebra over the field* $\bar{\Omega}_r = \Omega_r$. *If* Ω *is a void set, the ring is a simple zero algebra over the absolute. Conversely: if* $\mathfrak{H}$ *is a simple zero algebra over the field* $\mathfrak{K}$, *then is a* $\mathfrak{K}$-*simple zero ring with structure* $\mathfrak{H} = a\mathfrak{K}$, $(0 \neq a \in \mathfrak{H})$.

Let us consider now the Ω-simple rings $\mathfrak{S}$ which are not zero rings. As $\mathfrak{S}^2$ is an Ω-ideal, we have $\mathfrak{S}^2 = \mathfrak{S}$. The operator domain acts in a commutative way. Ω_r is a commutative ring and $\Omega_r \subseteq \bar{\mathfrak{T}} \cap \bar{\Omega}_r$. For every $0 \neq a \in \mathfrak{S}$, we have $a\mathfrak{T} \neq (0)$, because if $a\mathfrak{S}_r = (0) = a\mathfrak{S}$, there would be an admissible ideal $\mathfrak{a} \neq (0)$ such that $\mathfrak{a}\mathfrak{S} = (0)$, and consequently we would have $\mathfrak{S}^2 = (0)$. As $a\mathfrak{T} = \mathfrak{S}$, we see that $\mathfrak{S}$ is always a simple ring, with or without operators. $\mathfrak{T}$ is irreducible and its commutator $\bar{\mathfrak{T}}$ (a commutative ring) is a field. For no one $a \neq 0$ we will have $aE_\lambda = 0$, with $E_\lambda \neq 0$. We have then:

Theorem 19: *Let* $\mathfrak{S}$ *be a ring, not zero ring.* $\mathfrak{S}$ *is* Ω-*simple, if, and only if,* $\mathfrak{S}$ *is simple when considered without operators. An* Ω-*simple ring, not zero ring, is a simple algebra (not zero algebra) over its maximal operator domain, and, consequently, over any field which may be con-*

sidered an operator domain of the ring, if the unity acts as an unitary operator. Consequently:

Corollary 4: *Let $\mathfrak{A}$ be an algebra over $\mathfrak{K}$ (not zero algebra). $\mathfrak{A}$ is simple algebra, if, and only if, $\mathfrak{A}$ is a simple ring (not zero ring) without operators.*

Let us continue the same set of hypotheses on $\mathfrak{S}$. The center $\mathfrak{Z}$ is an operator domain. $c \neq 0$, $(c \in \mathfrak{Z})$, is not a zero divisor, because if $ca = 0$, $(a \neq 0)$, there would be a non null two sided annihilator of c, and consequently $c\mathfrak{S} = (0)$, which is impossible. $\mathfrak{Z}$ has then a faithful representation in $\overline{\mathfrak{T}}$ and it is a subfield of $\overline{\mathfrak{T}}$, as we can also see directly. For every $0 \neq c \in \mathfrak{Z}$, we have $c\mathfrak{S} = \mathfrak{S}$, and consequently the equation $cx = c'$, $(c' \in \mathfrak{Z})$, has a solution $x \in \mathfrak{S}$. We will show that $x \in \mathfrak{Z}$. Let us take some $a \in \mathfrak{S}$ and let y be such that $cy = a$. Then we have $xa = xcy = cxy = c'y = yc' = ycx = cyx = ax$, and $x \in \mathfrak{Z}$. The identity of $\mathfrak{Z}$, represented by the identity endomorphism of $\overline{\mathfrak{T}}$, is the identity of $\mathfrak{S}$. From the former theorem, we can give the following

Theorem 20: *Every Ω-simple ring, not zero ring, with a center $\mathfrak{Z} \neq (0)$, is a simple algebra over its center. The identity of $\mathfrak{X}$ is the identity of the ring.* We have also

Corollary 5: *An Ω-simple ring, not zero ring, has identity, if, and only if, its center $\mathfrak{Z}$ is $\neq (0)$.*

Let us study now the finite direct sums of simple rings, such that the products of two different summands are null. Those rings $\mathfrak{S}$ can be considered as generated by a finite number of simple ideals $\mathfrak{a}_i$. Let then be $\mathfrak{S} = \mathfrak{a}_1 + \cdots + \mathfrak{a}_t$, with $\mathfrak{a}_i \mathfrak{a}_j = (0)$, for $i \neq j$. In [(I), pgs. 26-27], we have given some propositions on this question, with some complements in [(I), pgs. 30-31], when $\mathfrak{S}$ has an identity. Independently of the existence of the identity, we can make the following remarks. Let us consider $\mathfrak{S}$ as a module over the operator domain $\mathfrak{T}$: $\mathfrak{S}$ will be completely reducible. For a two-sided ideal $\mathfrak{B}$ we can write $\mathfrak{S} = \mathfrak{B} + \mathfrak{C}$, where $\mathfrak{C} = \mathfrak{a}_{i_1} + \cdots + \mathfrak{a}_{i_r}$, with $\mathfrak{B} \cap \mathfrak{a}_{i_j} = (0)$, $(j = 1, 2, \cdots, r)$. Let $b = b_1 + \cdots + b_t$, $(b_k \in \mathfrak{a}_k)$, be the decomposition of $0 \neq b \in \mathfrak{B}$. For every $s = s_1 + \cdots + s_t \in \mathfrak{S}$, we have $bs = b_1 s_1 + \cdots + b_t s_t$, $sb = s_1 b_1 + \cdots + s_t b_t$. As $bs, sb \in \mathfrak{B}$, we see that, if $0 \neq b_k \in \mathfrak{a}_k$ belongs to the decomposition of $b \in \mathfrak{B}$, also $b_k s_k$, $s_k b_k$, $(s_k \in \mathfrak{a}_k)$, belong to the decomposition of another elements of $\mathfrak{B}$. If there is one $b_k \neq 0$, $\mathfrak{a}_k$ is the set of all b_k, because this set is a two--sided ideal of the simple ring $\mathfrak{a}_k$. We shall see now that, for each $(0) \neq \{b_k\} = \mathfrak{a}_k$, we have $\mathfrak{a}_k \subseteq \mathfrak{B}$. In fact, it is $\mathfrak{a}_k \mathfrak{B} = \mathfrak{a}_k^2 = \mathfrak{a}_k \subseteq \mathfrak{B}$. For the decomposition of $\mathfrak{S}$ given before, there are the summands $\mathfrak{a}_i \subseteq \mathfrak{B}$, whose direct sum gives $\mathfrak{B}$, and the others, for which $\mathfrak{a}_i \cap \mathfrak{B} = (0)$. We have then $\mathfrak{S} = \mathfrak{B} + \mathfrak{C}$, $\mathfrak{B} = \mathfrak{a}_{i_{r+1}} + \cdots + \mathfrak{a}_{i_t}$, $\mathfrak{C} = \mathfrak{a}_{i_1} + \cdots + \mathfrak{a}_{i_r}$. More shortly:

Theorem 21: *Let $\mathfrak{S}$ be a direct sum of a finite number of two-sided simple ideals, $\mathfrak{S} = \mathfrak{a}_1 + \cdots + \mathfrak{a}_t$. Then, for a two-sided ideal $\mathfrak{B}$, we have $\mathfrak{S} = \mathfrak{B} + \mathfrak{C}$, where $\mathfrak{B}$ is the sum of all $\mathfrak{a}_i$ such that $\mathfrak{B} \cap \mathfrak{a}_i = \mathfrak{a}_i$, and $\mathfrak{C}$ is the sum of those $\mathfrak{a}_j$ for which $\mathfrak{B} \cap \mathfrak{a}_j = (0)$,* [35].

Corollary 6: *On conditions of theorem 21, $\mathfrak{S}$ has an unique decomposition, or, as it is the same, the two-sided simple ideals of $\mathfrak{S}$ are only those of the unique decomposition.* Though this proposition seems less general than the one given in [(I), pgs. 30], we have to remember that there we have assumed the existence of identity in $\mathfrak{S}$.

7) **On non-associative rings** — A *non-associative ring (naring)* is a ring in which fails only the associativity of the product. It is, therefore, a module, whose elements are both right and left operators. Let $\mathfrak{X} = \{a, b, \cdots, r, s, \cdots,$

$x, y, \cdots\}$ be a naring. If there exists an operator domain Ω, we will suppose that the elements of Ω act on $\mathfrak{X}$, accordingly the properties 1), 2), 3) of § 4. In the absolute $\mathfrak{A}$, of $\mathfrak{X}$, we may also consider the associative subrings Ω_r, $\mathfrak{X}_r, \mathfrak{X}_l$, etc.. The subring $\mathfrak{E}(\mathfrak{X}_r, \mathfrak{X}_l)$ will be represented by $\mathfrak{Q}$. As in the associative case, we have the relations $\overline{\mathfrak{Q}} = \overline{\mathfrak{X}}_r \cap \overline{\mathfrak{X}}_l, \Omega_r \subseteq \overline{\mathfrak{Q}}$; and $\overline{\mathfrak{Q}}$ may be considered the maximal operator domain, because $\alpha \in \overline{\mathfrak{Q}}$ implies $(xy)\alpha = (xE_y^{(r)})\alpha = (x\alpha)E_y^{(r)} = (x\alpha)y = (yE_x^{(l)})\alpha = (y\alpha)E_x^{(l)} = x(y\alpha)$. In Jacobson, [4], $\mathfrak{Q}$ is called *multiplication ring of* $\mathfrak{X}$ and $\overline{\mathfrak{Q}}$ the *multiplication centralizer of* $\mathfrak{X}$. Here, we have, evidently, $\mathfrak{X}_r \not\subset \overline{\mathfrak{X}}_l$, $\mathfrak{X}_l \not\subset \overline{\mathfrak{X}}_r$, $\mathfrak{X}_r\mathfrak{X}_l \neq \mathfrak{X}_l\mathfrak{X}_r$; but $\overline{\mathfrak{Q}}\mathfrak{Q} = \mathfrak{Q}\overline{\mathfrak{Q}} \subseteq \mathfrak{Q}$, and, in particular, $\Omega_r\mathfrak{Q} = \mathfrak{Q}\Omega_r \subseteq \mathfrak{Q}$. In fact, we have $x \cdot E_a^{(r)}\alpha = (xa)\alpha = x \cdot a\alpha = xE_{a\alpha}^{(r)}$, $x \cdot E_a^{(l)}\alpha = (ax)\alpha = (a\alpha)x = xE_{a\alpha}^{(l)}$, and, by that, it is, more precisely, $\overline{\mathfrak{Q}}\mathfrak{X}_r \subseteq \mathfrak{X}_r$, $\overline{\mathfrak{Q}}\mathfrak{X}_l \subseteq \mathfrak{X}_l$. The elements $s \in \mathfrak{X}$, which may act as operators, are characterised by the property $E_s^{(r)} \in \overline{\mathfrak{Q}}$, and they form a subring $\mathfrak{B}$ of $\mathfrak{X}$. Really, let $s, s' \in \mathfrak{X}$ be such that $E_s^{(r)}$, $E_{s'}^{(r)} \in \overline{\mathfrak{Q}}$. Then, if $E_{ss'}^{(r)} = E_s^{(r)}E_{s'}^{(r)}$, we will have $E_{ss'}^{(r)} \in \overline{\mathfrak{Q}}$; but this is immediate, because $x \cdot ss' = (xs')s = (xs)s'$. Besides, we may write $E_s^{(r)}E_{s'}^{(r)} = E_{s'}^{(r)}E_s^{(r)}$. For the difference $s - s'$, the conclusion is the same. At last, $\mathfrak{B}$ is an associative ring, and its image $\mathfrak{B}_r \subseteq \overline{\mathfrak{Q}}$ is an associative and commutative ring. We have:

Theorem 22: *Let $\mathfrak{X}$ be a naring with operator domain Ω. In the absolute $\mathfrak{A}$, of its module, we have $\Omega_r \subseteq \overline{\mathfrak{X}}_r \cap \overline{\mathfrak{X}}_l = \overline{\mathfrak{Q}}$. $\overline{\mathfrak{Q}}$ is the maximal operator domain of $\mathfrak{X}$; $s \in \mathfrak{X}$ satisfies $E_s^{(r)} \in \overline{\mathfrak{Q}}$, if, and only if, s belongs to the associative subring $\mathfrak{B} \subseteq \mathfrak{X}$ characterized by the relations $(xy)s = (xs)y = x(ys)$, with arbitrary $x, y \in \mathfrak{X}$; at last, the image $\mathfrak{B}_r = \mathfrak{X}_0^{(r)} \cap \overline{\mathfrak{Q}} \subseteq \mathfrak{A}$, of $\mathfrak{B}$, is an associative and commutative ring.* [$\mathfrak{X}_0^{(r)}$ is the set of endomorphisms induced by the right multiplications by the elements of $\mathfrak{X}$].

We call *center* $\mathfrak{Z}$ of $\mathfrak{X}$ the set of elements of $\mathfrak{B}$ which commute with every element of $\mathfrak{X}$. It is an associative and commutative subring of $\mathfrak{B}$, and we have $\mathfrak{Z}_r \subseteq \mathfrak{B}_r \subseteq \overline{\mathfrak{Q}}$. We will see that $\mathfrak{Z}_r$ and $\mathfrak{B}_r$ are right ideals of $\overline{\mathfrak{Q}}$. At first, we have $\mathfrak{B}_r\overline{\mathfrak{Q}} \subseteq \mathfrak{X}_0^{(r)}\overline{\mathfrak{Q}} \subseteq \mathfrak{X}_0^{(r)}$, $\mathfrak{B}_r\overline{\mathfrak{Q}} \subseteq \overline{\mathfrak{Q}}$, and then $\mathfrak{B}_r\overline{\mathfrak{Q}} \subseteq \mathfrak{X}_r \cap \overline{\mathfrak{Q}} = \mathfrak{B}_r$. For $\mathfrak{Z}_r$, it is $\mathfrak{Z}_r\overline{\mathfrak{Q}} \subseteq \mathfrak{B}_r$; and if $c \in \mathfrak{Z}$, $\alpha \in \overline{\mathfrak{Q}}$, the relations $x \cdot c\alpha = xc \cdot \alpha = cx \cdot \alpha = c\alpha \cdot x$ prove that $c\alpha$ commutes with every element of $\mathfrak{X}$. We may also observe that the homomorphic correspondence $s \to E_s^{(r)}$, of $\mathfrak{B}$ onto $\mathfrak{B}_r$, is admissible with respect to $\overline{\mathfrak{Q}}$. It contains the correspondence $\mathfrak{Z} \to \mathfrak{Z}_r$. Then:

Theorem 23: *The center $\mathfrak{Z}$ of a naring $\mathfrak{X}$ is an associative and commutative subring of $\mathfrak{B}$, associative ring of theoreme 22. $\mathfrak{Z}_r$ and $\mathfrak{B}_r$ are right ideals of $\overline{\mathfrak{Q}}$, and the correspondences $\mathfrak{Z} \to \mathfrak{Z}_r$, $\mathfrak{B} \to \mathfrak{B}_r$ are $\overline{\mathfrak{Q}}$-homomorphisms.*

When a naring verifies $\mathfrak{X}^2 = \mathfrak{X}$, the centralizer $\overline{\mathfrak{Q}}$ is an associative commutative ring, as the ring $\overline{\mathfrak{C}}$ of § 4. The meaning of $\mathfrak{X}^2$ is the usual. We have:

Theorem 24: *If $\mathfrak{X}$ is a naring such that $\mathfrak{X}^2 = \mathfrak{X}$, every operator domain acts of commutative way. In particular, the multiplication centralizer is commutative.*

If $u \in \mathfrak{X}$ is an identity, $u \in \mathfrak{Z}$, evidently. Whatever may be the operator domain Ω, the application of λ or $u\lambda$ gives the same result; then $u\lambda \in \mathfrak{B}$. We have seen also that $u\lambda$ commutes with every element of $\mathfrak{X}$. Consequently $u\lambda \in \mathfrak{Z}$, and this signifies that the image $\mathfrak{Z}_r$ contains all the possible operator domains. The existence of u implies also the relations $\overline{\mathfrak{X}}_r \subseteq \mathfrak{X}_l$, $\overline{\mathfrak{X}}_l \subseteq \mathfrak{X}_r$; for example, if $B \in \overline{\mathfrak{X}}_l$, we have $u \to uB = b$, $x = xu \to xB = (xu)B = uE_x^{(l)}B = (uB)E_x^{(l)} = bE_x^{(l)} = xb = xE_b^{(r)}$, and then $B = E_b^{(r)}$. We cannot prove the equality $\mathfrak{X}_r = \overline{\mathfrak{X}}_l$, because, on the contrary of the associative case, it is not $\mathfrak{X}_r \subseteq \overline{\mathfrak{X}}_l$. We have the

THEOREM 25: *In a naring $\mathfrak{X}$ with identity, the center $\mathfrak{Z}$, maximal operator domain, has in the absolute the image* $\mathfrak{Z}_r = \mathfrak{G}_r = \overline{\mathfrak{Q}} = \overline{\mathfrak{X}}_r \cap \overline{\mathfrak{X}}_l \subseteq \mathfrak{X}_l \cap \mathfrak{X}_r$.

A last hypothesis, about Ω, is analogous to that one of § 4, which led to theorems 9 and 10. We may make the same assertions here. It is suitable to note that $\mathfrak{X} = \mathfrak{X}\varepsilon + \mathfrak{X}(1-\varepsilon)$ is also a decomposition in two-sided ideals.

The notion of admissible *right ideal* of $\mathfrak{X}$, generated by the element a, is the same as in the associative case. It is the set of elements $a\,\mathfrak{E}(1, \Omega_r, \mathfrak{X}_r)$. The *two-sided ideal* generated by a is the set $a\,\mathfrak{E}(1, \Omega_r, \mathfrak{Q})$. The expression $a\mathfrak{X}$ is not an admissible right ideal. The right ideal generated by $a\mathfrak{X}$ is $a\mathfrak{X}_r$, which is an admissible ideal. A right ideal $\mathfrak{r}$, of $\mathfrak{X}$, will be an Ω-subgroup which, with every $a \in \mathfrak{r}$ contains $a\,\mathfrak{E}(\Omega_r, \mathfrak{X}_r)$. The ideal $\mathfrak{r}^2$ may be defined as the set of elements of the form $\Sigma[aa'(\Sigma \pm 1 E_s^{(r)} \cdots E_t^{(r)})]$, where $a, a' \in \mathfrak{r}$ and the two $\Sigma\Sigma$ have a finite number of summands. It will be the right ideal generated by the elements aa. In this sense, we may define a product $\mathfrak{r}\mathfrak{r}'$, of two right ideals, as the right ideal generated by the elements rr', with $r \in \mathfrak{r}$, $r' \in \mathfrak{r}'$. It will be $\mathfrak{r}^2 \subseteq \mathfrak{r}$, $\mathfrak{r}\mathfrak{r}' \subseteq \mathfrak{r}$, but there will be a great dissymetry with the associative case, because the elements of $\mathfrak{r}\mathfrak{r}'$ cannot take the form $\Sigma rr'$. Opportunely, we had already interpreted a two-sided ideal $\mathfrak{a}$ as an Ω-subgroup which, with every $a \in \mathfrak{a}$, contains $a\,\mathfrak{E}(\Omega_r, \mathfrak{Q})$. If Ω is the void set, $\mathfrak{a}$ is a submodule with the property $\mathfrak{a}\mathfrak{Q} \subseteq \mathfrak{a}$. $\mathfrak{X}^2$ is always a two-sided ideal. It may be conceived as the set of elements of the form $\Sigma tt', (t, t' \in \mathfrak{X})$, whether Ω be void or not. If there exists $u \in \mathfrak{X}$, $a\,\mathfrak{E}(1, \Omega_r, \mathfrak{X}_r) = a\mathfrak{X}_r$ is the right ideal generated by a. The characteristic property of a right ideal $\mathfrak{r}$ is to be a sub-module, which, with every a, contains $a\mathfrak{X}_r$.

About idempotents, we limit our considerations to a simple remark. Let $f \in \mathfrak{X}$ be an idempotent. The left ideal $f\mathfrak{X}_l$, generated by f, is a $(\mathfrak{X}_l, \Omega_r)$-submodule. Its $\mathfrak{X}_l$-endomorphisms are $(\mathfrak{X}_l, \Omega_r)$-endomorphisms, because, given the $\mathfrak{X}_l$-endomorphism defined by the correspondence $f \to fA^{(l)}, (A^{(l)} \in \mathfrak{X}_l)$, we have $f = ff = fE_f^{(l)} \to fA^{(l)}E_f^{(l)} = fA^{(l)}$, $fE_\lambda = (ff)E_\lambda = fE_f^{(l)}E_\lambda = fE_{f\lambda}^{(l)} \to fA^{(l)}E_{f\lambda}^{(l)} = (fA^{(l)})E_f^{(l)}E_\lambda = fA^{(l)}E_\lambda$.

Let us consider now some details relative to simple narings. As they cannot be zero-rings, it will be $\mathfrak{X}^2 = \mathfrak{X}$. The multiplication centralizer $\overline{\mathfrak{Q}}$ is commutative. As, on the other hand, $\mathfrak{X}$ is $(\mathfrak{Q}, \Omega_r)$-simple, if we put $\mathfrak{P} = \mathfrak{E}(\mathfrak{Q}, \Omega_r)$, $\mathfrak{X}$ will be $\mathfrak{P}$-simple. Then $\overline{\mathfrak{P}} = \overline{\mathfrak{Q}} \cap \overline{\Omega}_r$ is a division ring, consequently a field, at the same time that, as $\overline{\mathfrak{Q}}$ is commutative, it will be also Ω_r commutative, and $\Omega_r \subseteq \overline{\mathfrak{P}}$. For every $0 \neq a \in \mathfrak{X}$, we have $a\mathfrak{Q} \neq (0)$, as in the associative case. Really, the hypothesis $a\mathfrak{Q} = (0)$ will imply, if $b\mathfrak{Q} = (0)$, the relation $(a-b)\mathfrak{Q} = (0)$, together with $(a\mathfrak{Q})\mathfrak{Q} = (0)$, $(a\Omega_r)\mathfrak{Q} = (a\mathfrak{Q})\Omega_r = (0)$. We would have $\mathfrak{X}\mathfrak{Q} = (0)$, and, in particular, $\mathfrak{X}^2 = (0)$. And we may conclude the equality $a\mathfrak{Q} = \mathfrak{X}$, because $a\mathfrak{Q} \neq (0)$ is an admissible two-sided ideal. The $(\mathfrak{Q}, \Omega_r)$-simplicity implies the $\mathfrak{Q}$-simplicity, and conversely. Then:

THEOREM 26: *If $\mathfrak{X}$ is an Ω-simple naring, it is also a simple naring without operators. An Ω simple naring is a non-associative algebra over every field, which may be considered an operator domain of the ring, if the unity of the field acts as unitary operator of $\mathfrak{X}$*. Consequently:

COROLLARY 7: *Let $\mathfrak{H}$ be a non-associative algebra over $\mathfrak{K}$. $\mathfrak{H}$ is a simple algebra, if, and only if, $\mathfrak{H}$ is a simple ring without operators.*

We have already said that the center $\mathfrak{Z}$ is an operator domain. If $0 \neq c \in \mathfrak{Z}$, we know that $c\mathfrak{Q} = \mathfrak{X}$. As $c\mathfrak{Q} = c\mathfrak{X}_r = c\mathfrak{X}_l = c\mathfrak{X}$, we have $c\mathfrak{X} = \mathfrak{X}$. Then, if we suppose that the center is $\neq (0)$, theorem 23 permits to write $\mathfrak{Z}_r = \overline{\mathfrak{Q}}$. In this case, $\mathfrak{Z}_r$ is a faithful representation of $\mathfrak{Z}$, and, consequently, the center is a field.

Besides, we may deduce the same in a direct way, as we have done for the associative rings. The reasonings which gave the relation $ax = xa$, for every solution of $cx = c'$, $(c, c' \in \mathfrak{Z})$, are valid. Next, the verification of $(ab)x = (ax)b = a(bx)$, with $a, b \in \mathfrak{X}$, may be carried on in the following way: writing $a = sc$, besides $xc = c'$, we have $(ab)x = (sc \cdot b)x = (sb \cdot c)x = sb \cdot xc = sb \cdot c' = sc' \cdot b = (s \cdot xc)b = (sc \cdot x)b = ax \cdot b$; and if we suppose $b = tc$, besides $cx = c'$, we see that $(ab)x = (a \cdot tc)x = (at \cdot c)x = at \cdot xc = at \cdot c' = a \cdot tc' = a \cdot (t \cdot xc) = a(tc \cdot x) = a \cdot bx$. The identity of $\mathfrak{Z}$, as is represented by the unitary endomorphism, is also the identity of $\mathfrak{X}$. We have:

Theorem 27: *Every Ω-simple naring $\mathfrak{X}$ with a center $\mathfrak{Z} \neq (0)$, is a simple algebra over its center. The identity of $\mathfrak{Z}$ is the identity of $\mathfrak{X}$. The image $\mathfrak{Z}_r$ is, then, the multiplication centralizer.*

Corollary 8: *An Ω-simple naring has identity, if, and only if, its center $\mathfrak{Z}$ is $\neq (0)$.*

We can prove directly that $1 \in \mathfrak{Z}$ is the identity of $\mathfrak{X}$: as $c\mathfrak{X} = \mathfrak{X}$, for every $0 \neq c \in \mathfrak{Z}$, we have $x = ct$, $1x = 1 \cdot ct = 1 \cdot tc = ct = x = x1$, $(x \in \mathfrak{X})$.

A last note on simple rings is the following one. If $\mathfrak{H}$ is a non-associative simple algebra, over $\mathfrak{K}$, the hypothesis $\mathfrak{Z} \neq (0)$ implies, by corollary 8, the existence of $1 \in \mathfrak{H}$. Then, the maximal operator domain of $\mathfrak{H}$ is its center,

and, thus, as the identity of $\mathfrak{K}$ is unitary operator, we can suppose $\mathfrak{K} \subseteq \mathfrak{H}$ a subfield of the field $\mathfrak{Z}$. Then:

Theorem 28: *Let $\mathfrak{H}$ be a non-associative simple algebra over $\mathfrak{K}$. $\mathfrak{H}$ has identity, if, and only if, its center $\mathfrak{Z}$ contains the field $\mathfrak{K}$.*

We will finish this § by the consideration of a naring $\mathfrak{X}$ which may be written in the form $\mathfrak{X} = \mathfrak{X}_1 + \cdots + \mathfrak{X}_t$, where the $\mathfrak{X}_i$ are simple narings, for which $x_i x_j = 0$, $(i \neq j, x_i \in \mathfrak{X}_i, x_j \in \mathfrak{X}_j)$. $\mathfrak{X}$ becomes decomposed in a sum of simple two-sided ideals, and as the proofs which led to theorem 21 and its corollary 6 are valid, we have the same propositions. In this case, as $\mathfrak{X}^2 \supseteq \mathfrak{X}_i^2 = \mathfrak{X}_i$, it is also $\mathfrak{X}^2 = \mathfrak{X}$. For the maximal operator domain of $\mathfrak{X}$, which is the multiplication centralizer $\overline{\mathfrak{Q}}$, we have the relations $\mathfrak{X}_i\overline{\mathfrak{Q}} \subseteq \mathfrak{X}_i$, as we conclude in sequel. It is $\mathfrak{X}_i\mathfrak{Q} \supseteq \mathfrak{X}_i^2 = \mathfrak{X}_i$, $\mathfrak{X}_i\mathfrak{Q} \subseteq \mathfrak{X}_i$, that is, $\mathfrak{X}_i\mathfrak{Q} = \mathfrak{X}_i$; on the other hand, as $\mathfrak{Q}\overline{\mathfrak{Q}} \subseteq \mathfrak{Q}$, we have $\mathfrak{X}_i\mathfrak{Q}\overline{\mathfrak{Q}} = \mathfrak{X}_i\overline{\mathfrak{Q}} \subseteq \mathfrak{X}_i\mathfrak{Q} = \mathfrak{X}_i$. We may state the following

Theorem 29: *Given a naring $\mathfrak{X}$, direct sum of a finite number of simple narings, of the form $\mathfrak{X} = \mathfrak{X}_1 + \cdots + \mathfrak{X}_t$, we can say:* 1) *$\mathfrak{X}$ does not contain two-sided simple ideals, besides the $\mathfrak{X}_i$:* 2) *for every two-sided ideal $\mathfrak{B}$, of $\mathfrak{X}$, we have always $\mathfrak{X} = \mathfrak{B} + \mathfrak{C}$, where $\mathfrak{B}$ is the sum of the $\mathfrak{X}_i$ for which $\mathfrak{B} \cap \mathfrak{X}_i = \mathfrak{X}_i$, and $\mathfrak{C}$ is the sum of the $\mathfrak{X}_j$ such that $\mathfrak{B} \cap \mathfrak{X}_j = (0)$;* 3) *every operator domain of $\mathfrak{X}$ acts in a commutative way, and it is an operator domain of every $\mathfrak{X}_i$,* [35].

8) **On the theory of the discrete direct sums** — In the following, M will be a set of elements $\alpha, \beta, \cdots, \lambda, \mu, \nu, \cdots$, every one being putted in correspondence with a module: $\mu \rightarrow \mathfrak{m}_\mu$. We will suppose that the operator domain, common

to the $\mathfrak{m}_\mu$, is a ring $\mathfrak{B}$, which may not be «immersed» in the several rings of endomorphisms, but has, in those rings, homomorphic images [(I), pgs. 231 and following].

Besides the discrete direct sum $\mathfrak{M} = \Sigma\, \mathfrak{m}_\nu, (\nu \in M)$, it will interest us the ring $\bar{\mathfrak{B}}$ of its $\mathfrak{B}$-endomorphisms. Let us take of $\bar{\mathfrak{B}}$ a set $\{A_\sigma\}$. This set is called *summable*, [9], if, for every $x \in \mathfrak{M}$, it is $xA_\sigma = 0$, except for a finite number of $A_\sigma A_\sigma$. Then we may consider ΣA_σ as a well determined endomorphism. Given $x \in \mathfrak{M}$, let us write $x = m_\lambda + + \cdots + m_\tau$, where $m_\lambda \in \mathfrak{m}_\lambda$, etc.. Every x is decomposed into a finite number of summands. The correspondence $x \to m_\lambda$ is an endomorphism $E_\lambda = E_{\lambda\lambda} \in \bar{\mathfrak{B}}$. Every endomorphism E_μ is idempotent, and the products $E_\mu E_\nu, (\mu \neq \nu)$, are null. The endomorphism $1 \in \bar{\mathfrak{B}}$ has the expression $1 = \Sigma E_\rho, (\rho \in M)$, precisely because $\{E_\rho\}$ is a summable set. When $\{A_\sigma\}$ is summable, hold the distributive equalities $(\Sigma A_\sigma) B = \Sigma A_\sigma B$, $B(\Sigma A_\sigma) = \Sigma BA_\sigma$. We may say, in the *only sense of sum of modules: the ring* $\bar{\mathfrak{B}}$, *of* $\mathfrak{B}$-*endomorphisms, of the discrete direct sum* $\mathfrak{M} = \Sigma\, \mathfrak{m}_\mu, (\mu \in M)$, *of modules* $\mathfrak{m}_\nu$ *(all* $\mathfrak{B}$*-modules), on the hypothesis* $1 = \Sigma E_\rho$, $(\rho \in M)$, *is complete direct sum of the right ideals* $E_\rho \bar{\mathfrak{B}}$. Every element $A \in \bar{\mathfrak{B}}$ appears as a sum of a summable set of the form $A = \sum\limits_\rho E_\rho A$, with $E_\rho A \in E_\rho \bar{\mathfrak{B}}$. On the other side, an expression of A, of the considered form, is unique; really, if we put $A = \Sigma E_\rho A_\rho, (A_\rho \in \bar{\mathfrak{B}})$, we define an endomorphism A such that $E_\lambda A = E_\lambda A_\lambda$, and then its representation is given by $A = \Sigma E_\rho A$.

Let us take an arbitrary endomorphism $S \in \bar{\mathfrak{B}}$. We have $S = (\sum\limits_\rho E_\rho S) \cdot \sum\limits_\sigma E_\sigma = \sum\limits_\rho [E_\rho S (\sum\limits_\sigma E_\sigma)] = \sum\limits_\rho (\sum\limits_\sigma E_\rho S E_\sigma) = = \sum\limits_{\rho,\sigma} E_\rho S E_\sigma$, noting that the two last expressions carry to the same result xS, when they are applied to every $x \in \mathfrak{M}$. Such result proceeds from a finite number of summands $E_\rho S E_\sigma$, exactly the same in the two expressions.

Putting $S_{\rho\sigma} = E_\rho S E_\sigma \in E_\rho \bar{\mathfrak{B}} E_\sigma$, we see that every endomorphism $S \in \bar{\mathfrak{B}}$ has an expression of the form $\sum\limits_{\rho,\sigma} S_{\rho\sigma}$, where the set $\{S_{\rho\sigma}\}$ is summable. Conversely, every summable set $\{S_{\rho\sigma}\}$ defines an endomorphism $S = \sum\limits_{\rho,\sigma} S_{\rho\sigma}$, with $E_\alpha S E_\beta = S_{\alpha\beta}$. We may say, in the *only sense of sum of modules: the ring* $\bar{\mathfrak{B}}$ *is special subdirect sum of the rings* $E_\alpha \bar{\mathfrak{B}} E_\beta = \bar{\mathfrak{B}}_{\alpha\beta}$, (*cfr.* [30, § 2]). *The elements of the subdirect sum are the summable sets of elements in the several* $\bar{\mathfrak{B}}_{\alpha\beta}$.

If we write $E_\rho S E_\sigma = E_\rho \cdot E_\rho S E_\sigma \cdot E_\sigma$ and note that $E_\rho S E_\sigma$ defines a well determined homomorphism $\mathfrak{m}_\rho \sim \mathfrak{m}'_\sigma \subseteq \subseteq \mathfrak{m}_\sigma$, which will be represented by $\sigma_{\rho\sigma}$, we see also that $S = \Sigma E_\rho \sigma_{\rho\sigma} E_\sigma, (\rho, \sigma \in M)$, where figure certain $\sigma_{\rho\sigma}$. Conversely, given a system of $\sigma_{\rho\sigma}$, we can prolong every homomorphism in order to form an endomorphism of $\mathfrak{M}$, namely $E_\rho \sigma_{\rho\sigma} E_\sigma$. If the set of these is summable, we obtain a well determined $S = \Sigma E_\rho \sigma_{\rho\sigma} E_\sigma$. Then:

Theorem 30: *There is a complete* 1—1 *correspondence between every* $S \in \bar{\mathfrak{B}}$ *and every system of homomorphisms* $\sigma_{\rho\sigma}$ *such that the set* $\{E_\rho \sigma_{\rho\sigma} E_\sigma\}$ *of endomorphisms of* $\mathfrak{M}$ *is summable.*

Clearly, we have $\bar{\mathfrak{B}}_{\alpha\beta} \bar{\mathfrak{B}}_{\beta\gamma} \subseteq \bar{\mathfrak{B}}_{\alpha\gamma}$, $\bar{\mathfrak{B}}_{\alpha\beta} \bar{\mathfrak{B}}_{\delta\gamma} = (0)$, if $\beta \neq \delta$. Putting $S = \Sigma S_{\rho\sigma}$, $T = \Sigma T_{\lambda\mu}$, it is $S + T = \Sigma (S_{\rho\sigma} + T_{\rho\sigma})$, as well as

$$ST = \sum_{\rho,\sigma} S_{\rho\sigma} \cdot \sum_{\lambda,\mu} T_{\lambda\mu} = \sum_{\rho,\sigma} (S_{\rho\sigma} \cdot \sum_{\lambda,\mu} T_{\lambda\mu}) = \sum_{\rho,\sigma} (\sum_{\lambda,\mu} S_{\rho\sigma} T_{\lambda\mu}) =$$
$$= \sum_{\rho,\sigma,\mu} S_{\rho\sigma} T_{\sigma\mu} = \sum_{\rho,\mu} R_{\rho\mu}, \quad \text{with} \quad R_{\rho\mu} = \sum_\sigma S_{\rho\sigma} T_{\sigma\mu}.$$

Though there is a complete 1—1 correspondence between the elements of $\bar{\mathfrak{B}}_{\alpha\beta}$ and the homomorphisms $\sigma_{\alpha\beta}$, determined by the relations $S_{\alpha\beta} \to \sigma_{\alpha\beta}$, such that $m_\alpha S_{\alpha\beta} =$

$= m_\alpha \sigma_{\alpha\beta}, (m_\alpha \in \mathfrak{m}_\alpha)$; $\sigma_{\alpha\beta} \to E_\alpha \sigma_{\alpha\beta} E_\beta = S_{\alpha\beta}$, we cannot say that $\mathfrak{P}_{\alpha\beta}$ is the ring of the homomorphisms $\mathfrak{m}_\alpha \sim \mathfrak{m}'_\beta \subseteq \mathfrak{m}_\beta$, because it has not sense to speak of the product of two such homomorphisms.

Let us consider, however, the ring $\bar{\mathfrak{P}}_{\alpha\alpha}$. Its interpretation, as ring of the $\mathfrak{P}$-endomorphisms of $\mathfrak{m}_\alpha$, results from the $1-1$ correspondence refered before, and the following considerations. We have

$$m_\alpha (S_{\alpha\alpha} + T_{\alpha\alpha}) = m_\alpha (\sigma_{\alpha\alpha} + \tau_{\alpha\alpha}),$$
$$m_\alpha S_{\alpha\alpha} T_{\alpha\alpha} = (m_\alpha S_{\alpha\alpha}) T_{\alpha\alpha} = (m_\alpha \sigma_{\alpha\alpha}) \tau_{\alpha\alpha} = m_\alpha \sigma_{\alpha\alpha} \tau_{\alpha\alpha},$$

and $S_{\alpha\alpha} = 0$, if $\sigma_{\alpha\alpha} = 0$. Then: given a discrete direct sum $\mathfrak{M} = \Sigma\, \mathfrak{m}_\mu$, the ring $\bar{\mathfrak{P}}_{\alpha\alpha} = E_\alpha \bar{\mathfrak{P}} E_\alpha$ and the ring of the $\mathfrak{P}$-endomorphisms of the submodule $\mathfrak{m}_\alpha$ are isomorphic.

Let us suppose, next, that, in the decomposition $\mathfrak{M} = \Sigma\, \mathfrak{m}_\mu$, all the submodules are $\mathfrak{P}$-isomorphic to a fixed module $\mathfrak{m}$. Holds the following proposition, [9]: given the discrete direct sum $\mathfrak{M} = \Sigma\, \mathfrak{m}_\mu, (\mu \in M)$, of $\mathfrak{P}$-modules, $\mathfrak{P}$-isomorphic to a fixed module $\mathfrak{m}$, the ring $\bar{\mathfrak{P}}$, of the $\mathfrak{P}$-endomorphisms of $\mathfrak{M}$, is isomorphic to the ring of all the (transfinite) matrices, with M dimensions, formed by summable rows of $\mathfrak{P}$-endomorphisms belonging to the commutator $\bar{\mathfrak{P}}'$ of the image of $\mathfrak{P}$ in the ring of endomorphisms of $\mathfrak{m}$. For the demonstration, we will treat four preparatory questions.

In the first place: the sum and the product of two matrices of summable rows, with elements of $\bar{\mathfrak{P}}'$, are matrices of summable rows with elements of $\bar{\mathfrak{P}}'$. The proposition relative to the sum is trivial. We will treat only the case of the product. An element of the product

is of the form $\gamma_{\mu\nu} = \Sigma\, \alpha_{\mu\rho} \beta_{\rho\nu}, (\mu, \nu$ fixed; $\alpha_{\mu\rho}, \beta_{\rho\nu} \in \bar{\mathfrak{P}}')$. The sum is well defined, because, taking $\xi \in \mathfrak{m}$, in the μ-row there is only a finite number of values of ρ such that $\xi \alpha_{\mu\rho} \neq 0$. Then it is also necessary to use only a finite number of $\beta_{\rho\nu}$, and the application of $\gamma_{\mu\rho}$ to the element ξ is well determined. Let us study, next, the μ-row of the product. The question is to see that the set of the $\gamma_{\mu\nu}$, (μ fixed, $\nu \in M$), is summable. Let us take again $\xi \in \mathfrak{m}$. To every $\nu \in M$, corresponds, as we have said already, a sum

$$\alpha_{\mu a} \beta_{a\nu} + \cdots + \alpha_{\mu q} \beta_{q\nu}, \quad (a, \cdots, q \in M \text{ fixed}). \tag{1}$$

It matters to verify that, if ν becomes different, only a finite member of the former sums does not carry to zero, when applied to ξ. Let us write the different sums (1):

$$\alpha_{\mu a} \beta_{a\lambda} + \cdots + \alpha_{\mu q} \beta_{q\lambda}; \quad \alpha_{\mu a} \beta_{a\sigma} + \cdots + \alpha_{\mu q} \beta_{q\sigma}; \cdots. \tag{2}$$

Considered the element $\xi \alpha_{\mu a} \in \mathfrak{m}$, only a finite number of elements $\beta_{a\lambda}, \beta_{a\sigma}, \cdots$ may carry to $\xi \alpha_{\mu a} \beta_{a\lambda} \neq 0; \cdots$; the same holds to $\xi \alpha_{\mu b}, \cdots, \xi \alpha_{\mu q}$. To the last one, only a finite number of elements $\beta_{q\lambda}, \beta_{q\sigma}, \cdots$ may carry to $\xi \alpha_{\mu q} \beta_{q\lambda} \neq 0; \cdots$. Then, only a finite number of sums (2) annuls ξ, and the first question is treated.

In the second place: on the conditions of the proposition to prove, the ring $\bar{\mathfrak{P}}$ has a system of unity matrices $E_{\lambda\mu}$, for which, by definition, $E_{\lambda\rho} E_{\rho\mu} = E_{\lambda\mu}$, $E_{\lambda\rho} E_{\sigma\mu} = 0$, if $\rho \neq \sigma$. We have seen already that there are endomorphisms $E_\rho = E_{\rho\rho}$ such that $1 = \Sigma\, E_\rho$. The remaining unity matrices are formed in the following way. Let us represent by φ_λ the isomorphism $\mathfrak{m} \simeq \mathfrak{m}_\lambda = \mathfrak{m} \varphi_\lambda$. The isomorphism $\mathfrak{m}_\lambda \simeq \mathfrak{m}_\sigma$ results from the relations $\mathfrak{m}_\lambda \varphi_\lambda^{-1} = \mathfrak{m}$, $\mathfrak{m} \varphi_\sigma = \mathfrak{m}_\sigma = \mathfrak{m}_\lambda \varphi_\lambda^{-1} \varphi_\sigma$. We will put, therefore, $\Delta_{\lambda\sigma} = \varphi_\lambda^{-1} \varphi_\sigma$, for representing the isomorphism, well determined, which carries from $\mathfrak{m}_\lambda$ to $\mathfrak{m}_\sigma$. Next, it will be $E_\lambda \Delta_{\lambda\sigma} = E_{\lambda\sigma}$. We see immediately that $E_{\lambda\rho} E_{\rho\mu} = E_{\lambda\mu}$, $E_{\lambda\rho} E_{\sigma\mu} = 0$, as we have said. $E_\lambda \Delta_{\lambda\sigma}$ gives a particular example of the extension of a homomorphism $\mathfrak{m}_\lambda \sim \mathfrak{m}'_\rho$, represented before by $\sigma_{\lambda\sigma}$. And the second question is so finished.

In the third place: on the conditions of the proposition to prove, if we write, as in the general case, $S = \Sigma S_{\mu\nu}$, we have $S_{\mu\nu} = E_{\mu\nu} S'_{\mu\nu}$, with $S'_{\mu\nu} = \sum_\varkappa E_{\varkappa\mu} S E_{\nu\varkappa}$. It is immediate that $S'_{\mu\nu}$ has sense, with the definition given before. Next, it is $E_{\mu\nu} S'_{\mu\nu} = \sum_\varkappa E_{\mu\nu} E_{\varkappa\mu} S E_{\nu\varkappa} = E_{\mu\mu} S E_{\nu\nu} = S_{\mu\nu}$, and the third question is treated.

In the fourth place: the set of the elements $S'_{\mu\nu}$ forms a ring independent of the μ and ν, which is isomorphic to $\overline{\mathfrak{R}}'$ or to $E_\alpha \overline{\mathfrak{R}} E_\alpha$, whatever α may be. Fixed α, let us make to correspond $\sum_\varkappa E_{\varkappa\sigma} A E_{\alpha\varkappa} = A'$ to every element $A \in \overline{\mathfrak{R}}_{\alpha\alpha}$. We see that, fixed μ and ν, we have

$$\sum_\varkappa E_{\varkappa\alpha} A E_{\alpha\varkappa} = \sum_\varkappa E_{\varkappa\mu} (E_{\mu\alpha} A E_{\alpha\nu}) E_{\nu\varkappa}. \qquad (3)$$

The first member is independent of μ and ν, and the second shows that it is an element $S'_{\mu\nu}$. Conversely, given an element $S'_{\mu\nu}$, we have always

$$S'_{\mu\nu} = \sum_\varkappa E_{\varkappa\mu} S E_{\nu\varkappa} = \sum_\varkappa E_{\varkappa\alpha} \cdot E_{\alpha\alpha} (E_{\alpha\mu} S E_{\nu\alpha}) E_{\alpha\alpha} \cdot E_{\alpha\varkappa},$$

whence we conclude that $S'_{\mu\nu}$ has the form of the first member of (3), that is, it corresponds to an element $A = E_{\alpha\alpha} (E_{\alpha\mu} S E_{\nu\alpha}) E_{\alpha\alpha}$. This correspondence is $1-1$, because the equation in A,

$$\sum_\varkappa E_{\varkappa\mu} S E_{\nu\varkappa} = \sum_\sigma E_{\sigma\alpha} A E_{\alpha\sigma}, \qquad (A \in \overline{\mathfrak{R}}_{\alpha\alpha}),$$

has a well determined solution. Indeed, if we multiply the two members by $E_{\alpha\alpha}$, at the right or at the left, we obtain $E_{\alpha\mu} S E_{\nu\alpha} = E_{\alpha\alpha} A E_{\alpha\alpha} = A$. Denoting by $\mathfrak{A}'$ the set of the $S'_{\mu\nu}$, it remains to prove the ring-isomorphism $\overline{\mathfrak{R}}_{\alpha\alpha} \simeq \mathfrak{A}'$. For example, if $A, B \in \overline{\mathfrak{R}}_{\alpha\alpha}$, we have $\sum_\varkappa E_{\varkappa\alpha} A B E_{\alpha\varkappa} = \sum_\varkappa E_{\varkappa\alpha} A E_{\alpha\varkappa} \cdot \sum_\sigma E_{\sigma\alpha} B E_{\alpha\sigma}$, and the assertion is immediate.

At last, let us consider the proposition. As every $S \in \overline{\mathfrak{R}}$ has the form $S = \Sigma E_{\mu\nu} S'_{\mu\nu}$, $(\mu, \nu \in M)$, and the ring $\mathfrak{A}'$, of the $S'_{\mu\nu}$, is independent of μ and ν, we conclude that $\overline{\mathfrak{R}}$ is a complete ring of matrices with elements of $\mathfrak{A}'$. If we consider a row of elements $S'_{\mu\nu}$, (μ fixed; $\nu \in M$), the corresponding ones in $E_\alpha \overline{\mathfrak{R}} E_\alpha \simeq \overline{\mathfrak{R}}'$ are of the form $A = E_{\alpha\alpha} (E_{\alpha\mu} S E_{\nu\alpha}) E_{\alpha\alpha} = E_{\alpha\mu} S E_{\nu\alpha}$. It remains to prove that, for every $m_\alpha \in \mathfrak{m}_\alpha$, only a finite number of the former AA carries to a result $\neq 0$. Now, as m_α is fixed, $m_\alpha E_{\alpha\mu} S$ is determined. If we put $m_\alpha E_{\alpha\mu} S = m'_\rho + \cdots + m'_\sigma$, it is enough to consider, in the expressions of the AA, the values $\nu = \rho, \cdots, \sigma$, as we want. The proposition is proved.

Constructed $\overline{\mathfrak{R}}$ and $\overline{\mathfrak{R}}'$, the modules $\mathfrak{M}$ and $\mathfrak{m}$ admit, respectively, those rings as operator domains. They are closed domains, in the sense defined in § 3.

Here it is an important proposition:

Theorem 31: *Given a ring $\mathfrak{R}_1$, let $\mathfrak{m}$ be a fixed $\mathfrak{R}_1$-module, and let us suppose $\mathfrak{M} = \Sigma \mathfrak{m}_\mu$, $(\mu \in M)$, a discrete direct sum (finite or infinite) of modules $\mathfrak{m}_\mu$, $\mathfrak{R}_1$-isomorphic to $\mathfrak{m}$. If $\overline{\mathfrak{R}}$ is the commutator of the image $\mathfrak{R}$, of $\mathfrak{R}_1$, in the absolute of $\mathfrak{M}$, the ring $\overline{\overline{\mathfrak{R}}}$, of the $\overline{\mathfrak{R}}$-endomorphisms of $\mathfrak{M}$, or commutator of $\overline{\mathfrak{R}}$ in that absolute, has always the same structure, namely: the structure of the ring $\overline{\overline{\mathfrak{R}}}'$, commutator of* *the commutator* $\bar{\mathfrak{R}}$ *of the image* $\mathfrak{S}$, *of* $\mathfrak{R}_1$, *in the absolute* $\mathfrak{a}$, *of* $\mathfrak{m}$. If $\Theta \in \bar{\mathfrak{R}}$, let us study its application to $\mathfrak{m}_\mu$. It must be $m_\mu \to m_\mu \Theta$, $m_\mu = m_\mu E_{\mu\mu} \to m_\mu E_{\mu\mu} \Theta = m_\mu \Theta E_{\mu\mu} = m_\mu \Theta \in \mathfrak{m}_\mu$, because Θ commutes, in particular, with all the $E_{\mu\nu}$. Then Θ, within $\mathfrak{m}_\mu$, is an endomorphism θ_μ. Accordingly reasonings of § 2, in the isomorphism $\mathfrak{a} \simeq \mathfrak{a}_\mu$, of the absolutes of $\mathfrak{m}$ and $\mathfrak{m}_\mu$, we will have $\theta_\mu \to \theta \in \mathfrak{a}$. As $\bar{\mathfrak{S}}_\mu$, commutator of the image $\mathfrak{S}_\mu$, of $\mathfrak{R}_1$, in $\mathfrak{a}_\mu$, may be written $\bar{\mathfrak{S}}_\mu = E_\mu \bar{\mathfrak{R}} E_\mu$, and as Θ commutes with every $E_\mu S E_\mu$, it follows that θ_μ commutes with these elements, and $\theta_\mu \in \bar{\mathfrak{S}}_\mu$. It will be $\theta \in \bar{\mathfrak{S}}$. In this way, to every $\Theta \in \bar{\mathfrak{R}}$ corresponds a well determined element $\theta \in \bar{\mathfrak{S}}$. The element θ is independent of the index μ, as we will conclude, noting that, if the isomorphism $\mathfrak{m}_\mu \simeq \mathfrak{m}_\nu$ carries m_μ into m_ν, it carries also $m_\mu \Theta$ into $m_\nu \Theta$. Now we have $m_\mu \Theta \to (m_\mu \Theta) \Delta_{\mu\nu} = (m_\mu \Theta) E_\mu \Delta_{\mu\nu} = (m_\mu \Theta) E_{\mu\nu} = (m_\mu E_{\mu\nu}) \Theta = (m_\mu \Delta_{\mu\nu}) \Theta = m_\nu \Theta$. As for the isomorphism $\bar{\mathfrak{R}} \simeq \bar{\mathfrak{S}}$, the conclusion is now immediate.

Corollary 9: *If* Θ *is a* $\bar{\mathfrak{R}}$*-endomorphism which applies a submodule* $\mathfrak{m}_\mu$ *into* (0), *then* $\Theta = 0$.

Corollary 10: *If* $\mathfrak{m}$ *is a* $\mathfrak{R}_1$*-module,* $\mathfrak{R}_1$*-closed, every discrete direct sum of modules* $\mathfrak{R}_1$*-isomorphic to* $\mathfrak{m}$ *is a* $\mathfrak{R}_1$*-module,* $\mathfrak{R}_1$*-closed.*

We will finish this § with a proof of the following theorem, which simplifies that one of [24, § 12, theorem 53].

Theorem 32: *In the sum* $\mathfrak{M} = \Sigma\, \mathfrak{m}_\mu$, *of modules isomorphic, refered in theorem 31, there is a complete* 1—1 *correspondence between the* $\bar{\mathfrak{R}}$*-submodules of* $\mathfrak{M}$ *and the* $\bar{\mathfrak{S}}$*-submodules* $\mathfrak{m}$. Let $\mathfrak{N}$ be a $\bar{\mathfrak{R}}$-submodule of $\mathfrak{M}$. By the homomorphisms $\mathfrak{M} \sim \mathfrak{m}_\mu$, are defined homomorphisms $\mathfrak{N} \sim \mathfrak{n}_\mu = \mathfrak{N} E_\mu$.

We see that $\mathfrak{n}_\mu \subseteq \mathfrak{N}$. Then $\mathfrak{N} = \Sigma\, \mathfrak{n}_\mu$. By $\mathfrak{n}_\mu$, we define $\mathfrak{n}_\mu \varphi_\mu^{-1} = \mathfrak{n} \subseteq \mathfrak{m}$. We will recognize the two following properties of $\mathfrak{n}$: 1) $\mathfrak{n}$ is $\bar{\mathfrak{S}}$-submodule; 2) $\mathfrak{n}$ is independent of the index μ. Let us make $\mu = \alpha$. To prove 1), it is enough to verify that $\mathfrak{n}_\alpha E_\alpha \bar{\mathfrak{R}} E_\alpha \subseteq \mathfrak{n}_\alpha$, which is immediate because $\mathfrak{n}_\alpha E_\alpha \bar{\mathfrak{R}} E_\alpha = \mathfrak{N} E_\alpha \cdot E_\alpha \bar{\mathfrak{R}} E_\alpha = \mathfrak{N} E_\alpha \bar{\mathfrak{R}} E_\alpha \subseteq \mathfrak{N} E_\alpha = \mathfrak{n}_\alpha$. With respect to 2), we must verify the equality $\mathfrak{N} E_\mu \varphi_\mu^{-1} = \mathfrak{N} E_\nu \varphi_\nu^{-1}$, that is, $\mathfrak{N} E_\mu \varphi_\mu^{-1} \varphi_\nu = \mathfrak{N} E_\nu$. The first member represents $\mathfrak{N} E_\mu \Delta_{\mu\nu} = \mathfrak{N} E_{\mu\nu} = \mathfrak{N} E_{\mu\nu} E_{\nu\nu} \subseteq \mathfrak{N} E_\nu$. As, in the same way, $\mathfrak{N} E_{\nu\mu} \subseteq \mathfrak{N} E_\nu$, it is $\mathfrak{N} E_{\nu\mu} E_{\mu\nu} \subseteq \mathfrak{N} E_{\mu\mu} E_{\mu\nu}$, that is, $\mathfrak{N} E_\nu \subseteq \mathfrak{N} E_{\mu\nu}$. Therefore, $\mathfrak{N} E_{\mu\nu} = \mathfrak{N} E_\nu$, as we want. Conversely, let us take $\mathfrak{n}$, supposed $\bar{\mathfrak{S}}$-submodule. We have $\mathfrak{n} \to \mathfrak{n}_\mu = \mathfrak{n} \varphi_\mu$, $\mathfrak{n} \to \mathfrak{n}_\nu = \mathfrak{n} \varphi_\nu = \mathfrak{n}_\mu \varphi_\mu^{-1} \varphi_\nu = \mathfrak{n}_\mu \Delta_{\mu\nu} = \mathfrak{n}_\mu E_\mu \Delta_{\mu\nu} = \mathfrak{n}_\mu E_{\mu\nu}$. Next, let us construct $\mathfrak{N} = \Sigma\, \mathfrak{n}_\mu$. The question is to see that $\mathfrak{N}$ is $\bar{\mathfrak{R}}$-submodule. Let us take $S = \Sigma E_\mu S E_\nu \in \bar{\mathfrak{R}}$. For applying S to $\mathfrak{N}$, we have to apply S to every $\mathfrak{n}_\mu$. But, then, it is enough to apply to $\mathfrak{n}_\mu$ the several $E_\mu S E_\nu$, (μ fixed, ν arbitrary). We have $\mathfrak{n}_\mu E_\mu S E_\nu = \mathfrak{n}_\mu E_\mu S E_\mu \cdot E_{\mu\nu} \subseteq \mathfrak{n}_\mu E_{\mu\nu} = \mathfrak{n}_\nu$. We see that we do not go out from $\mathfrak{N}$ when we apply to it any $S \in \bar{\mathfrak{R}}$. The theorem is now immediate.

9) **On semi-simple modules** —We will begin this § by some different proofs of lemma 17, theorem 38 and corollary 4 of [30, pgs. 146-148]. They are propositions on semi-simple modules $\mathfrak{M}$ with the ring of operators $\mathfrak{S}$.

Let C be the set of the simple submodules of the semi-simple module $\mathfrak{M}$: $C = \{\mathfrak{m}_\alpha, \mathfrak{m}_\beta, \cdots, \mathfrak{m}_\lambda, \cdots\}$. Let S be the set of all direct discrete sums of the submodules of C. It will be $S = \{\mathfrak{m}_\alpha, \mathfrak{m}_\beta, \cdots, \mathfrak{m}_\lambda, \cdots, \mathfrak{m}_\alpha + \mathfrak{m}_\beta, \cdots, \Sigma\, \mathfrak{m}_\lambda, \cdots\}$ in which we suppose that $\mathfrak{m}_\alpha \neq \mathfrak{m}_\beta$. The set S is a partially ordered set. Let T be an ordered subset of S. There exists the join element of T which is the direct discrete sum of all $\mathfrak{m}_\mu$ which belong to the sums of T. [It is easily seen that in the maximal element of T, as in every element of T, if the finite sum $m_\gamma + m_\delta + \cdots + m_\lambda = 0$, $m_\gamma \in \mathfrak{m}_\gamma$, etc., we have $m_\gamma = m_\delta = \cdots = m_\lambda = 0$]. T is then an inductive set [23, § 5], and by ZORN's ***principle*** there exists a maximal element $\Sigma \mathfrak{m}_\nu$ in S. And, consequently, we have $\mathfrak{M} = \Sigma \mathfrak{m}_\nu$, because, if we could have $\mathfrak{m}_\alpha \not\subset \Sigma \mathfrak{m}_\nu$, the direct discrete sum $\mathfrak{m}_\alpha + \Sigma \mathfrak{m}_\nu$ will contain $\Sigma \mathfrak{m}_\nu$, which, consequently, would not be a maximal element. Conversely, if $\mathfrak{M} = \Sigma \mathfrak{m}_\nu$ (as a direct discrete sum), where the $\mathfrak{m}_\nu$ are simple, $\mathfrak{M}$ is semi-simple. And the already referred lemma 17 is the following: $\mathfrak{M}$ is semi-simple if, and only if, we can give to $\mathfrak{M}$ the form $\mathfrak{M} = \Sigma \mathfrak{m}_\nu (\nu \in M)$, as a direct discrete sum of simple submodules $\mathfrak{m}_\nu$.

With respect to the theorem 38, let us suppose $\mathfrak{M}$ a semi-simple module. Let $\mathfrak{N} \neq \mathfrak{M}$ be a submodule of $\mathfrak{M}$ and let us consider the simple submodules $\mathfrak{m}_\alpha, \mathfrak{m}_\beta, \cdots$, of $\mathfrak{M}$, not contained in $\mathfrak{N}$. Clearly, $\mathfrak{M}$ is generated by $\mathfrak{N}$ and by the submodules. Let us take S as the following set of discrete direct sums: $S = \{\mathfrak{N}, \mathfrak{N} + \mathfrak{m}_\alpha, \cdots, \mathfrak{N} + \Sigma \mathfrak{m}_\lambda, \cdots\}$. Similarly to the proof of the preceding lemma, we have a maximal element in S, such that $\mathfrak{M} = \mathfrak{N} + \Sigma \mathfrak{m}_\lambda$, and we have $\mathfrak{M} = \mathfrak{N} + \mathfrak{N}'$ with $\mathfrak{N}' = \Sigma \mathfrak{m}_\lambda$. Conversely, let us suppose that for every $\mathfrak{N}$ we have a $\mathfrak{N}'$ such that $\mathfrak{M} = \mathfrak{N} + \mathfrak{N}'$. As $\mathfrak{M}$ is isomorphic to a subdirect sum of subdirectly irreducible modules $\mathfrak{m}_\mu$, by the homomorphism $\mathfrak{M} \sim \mathfrak{m}_\mu$ we have $\mathfrak{m}_\mu \simeq \mathfrak{M}/\mathfrak{M}_\mu$ and as for $\mathfrak{M}_\mu$ we have $\mathfrak{M}'_\mu$ such that $\mathfrak{M} = \mathfrak{M}_\mu + \mathfrak{M}'_\mu$. If, for every μ, $\mathfrak{M}'_\mu = (0)$, we have $\mathfrak{M} = \mathfrak{M}_\mu$, $\mathfrak{m}_\mu = (0)$, $\mathfrak{M} = (0)$. If $\mathfrak{M} \neq (0)$, there are some $\mathfrak{M}'_\mu \neq (0)$ subdirectly irreducible and, as they have the same property of $\mathfrak{M}$, the $\mathfrak{M}'_\mu$ are simple. Let us consider the direct discrete sum $\mathfrak{M}'$ of all simple submodules of $\mathfrak{M}$. We have $\mathfrak{M} = \mathfrak{M}' + \mathfrak{M}''$. We will show $\mathfrak{M}''$ is the null module. As $\mathfrak{M}''$ has the same property of $\mathfrak{M}$, $\mathfrak{M}''$ is the null module or $\mathfrak{M}''$ contains simple submodules which is absurd. Hence we have: $\mathfrak{M}$ is semi-simple if, and only if, for every submodule $\mathfrak{N}$, we can find a submodule $\mathfrak{N}'$ such that $\mathfrak{M} = \mathfrak{N} + \mathfrak{N}'$.

To corollary 4 of [30] we can give the following form: let be given to the semi-simple module $\mathfrak{M}$ the form $\mathfrak{M} = \Sigma \mathfrak{m}_\nu$, where the $\mathfrak{m}_\nu$ are simple; then, in the decomposition for $\mathfrak{N}$, $\mathfrak{M} = \mathfrak{N} + \mathfrak{N}'$, we can write $\mathfrak{N}'$ as a direct discrete sum of some $\mathfrak{m}_\nu$. The proof can be carried away as in the former propositions, using only those $\mathfrak{m}_\nu$, not contained in $\mathfrak{N}$, which belong to the decomposition $\Sigma \mathfrak{m}_\nu$.

Let be $\mathfrak{M} = \Sigma \mathfrak{m}_\nu, (\nu \in M)$, and $\mathfrak{M} = \Sigma \mathfrak{n}_j, (j \in N)$, two decompositions of $\mathfrak{M}$ as direct discrete sums of simple modules $\mathfrak{m}_\mu$ and $\mathfrak{n}_j$. We will prove that: the sets M and N have the same cardinality, [32], [33]. The proof is the following one: if one of the sets (M or N) is a finite one, the same hapens to the other and they have the same cardinal. Let us then suppose that the two sets are infinite and that $\mathfrak{M}$ is an $\mathfrak{S}$-module. Then we have $\mathfrak{m}_\nu = \{m e_\nu + e_\nu \mathfrak{S}\}$, $\mathfrak{n}_j = \{n f_j + f_j \mathfrak{S}\}$, where m, n are integers and $0 \neq e_\nu \in \mathfrak{m}_\nu$, $0 \neq f_j \in \mathfrak{n}_j$. For e_λ we have the decomposition $e_\lambda = n_i f_i + \cdots + n_k f_k + f_i s_i + \cdots + f_k s_k$, where the nn are integers and the ss operators of $\mathfrak{S}$. In the decomposition of all e_λ, we use all f_j, because, f_q does not belong to the decompositions of all e_λ, as we have $f_q = m_\alpha e_\alpha + \cdots + m_\rho e_\rho + e_\alpha t_\alpha + \cdots + e_\rho t_\rho$, where the mm are integers and the tt operators of $\mathfrak{S}$, the substitution of the decompositions of the $e_\alpha, \cdots, e_\rho$ in the f_q shows that the sum $\Sigma \mathfrak{n}_j$ would not be a direct discrete one. Given $j \in N$, we take f_j and the e_ν whose decompositions contain f_j. Then, for every j, we can consider the set $\{\lambda, \mu, \cdots\} \subset M$ of the indices ν of the referred e_ν. By the ZERMELO's ***axiom***, we can obtain a selector Θ that for each j

gives an $\nu = \Theta(j) \in M$. Then $\Theta(N) = M \subset M$. Let us consider now the function $j = \Theta^{-1}(\nu)$. For each ν, we can obtain some j such that $\Theta(j) = \nu$, but it is easily seen that the number of those j is a finite one. We have then obtained a $1-1$ correspondence between the elements of M' and the finite disjoint subsets $\Theta^{-1}(\nu) \subset N$. As the cardinality of the set of the $\Theta^{-1}(\nu)$ is the same as the one of N, we see that M' and N have the same cardinality. Changing the roles of M and N, we see also that M and a subset $N' \subset N$ have the same cardinality. Then M and N have the same cardinality.

The following theorem is an useful one, when used under more restrictive conditions:

Theorem 33: *Let $\mathfrak{M}$ be a semi-simple module and let $\mathfrak{M}_1$ be an $\mathfrak{S}$-submodule such that $\mathfrak{M}_1 A_j \subseteq \mathfrak{M}_1$, for the n $\mathfrak{S}$-endomorphisms $A_1, \cdots, A_n$; then we can write $\mathfrak{M}_1 = \mathfrak{Q}_1 + \mathfrak{N}_1$, where $\mathfrak{Q}_1$ is annihilated by the A_j and $\mathfrak{N}_1 A_j \subseteq \mathfrak{N}_1$.*
Let be $\mathfrak{N}_1'$ the kernel of the endomorphism $\mathfrak{M}_1 \sim \mathfrak{M}_1 A_1$. We have $\mathfrak{M}_1 = \mathfrak{N}_1' + \mathfrak{N}_1''$. For $\mathfrak{P}_1 = (\mathfrak{N}_1'', \mathfrak{N}_1'' A_1)$, we have $\mathfrak{P}_1 A_1 \subseteq \mathfrak{P}_1$, because $\mathfrak{M}_1 A_1 = \mathfrak{N}_1'' A_1$, $\mathfrak{M}_1 A_1 A_1 \subseteq \mathfrak{M}_1 A_1$, $\mathfrak{N}_1'' A_1 A_1 \subseteq \mathfrak{M}_1 A_1 = \mathfrak{N}_1'' A_1$. But, as $\mathfrak{P}_1 \supseteq \mathfrak{N}_1''$, we have $\mathfrak{P}_1 = \mathfrak{P}_1 \cap \mathfrak{N}_1' + \mathfrak{N}_1''$, and, as $\mathfrak{N}_1' = \mathfrak{P}_1 \cap \mathfrak{N}_1' + \mathfrak{O}_1$, we obtain $\mathfrak{M}_1 = (\mathfrak{P}_1 \cap \mathfrak{N}_1') + \mathfrak{O}_1 + \mathfrak{N}_1'' = \mathfrak{P}_1 + \mathfrak{O}_1$, with $\mathfrak{O}_1 \subseteq \mathfrak{N}_1'$, $\mathfrak{O}_1 A_1 = (0)$. Then, if $n = 1$, we have $\mathfrak{Q}_1 = \mathfrak{O}_1$, $\mathfrak{N}_1 = \mathfrak{P}_1$. We will continue the proof by induction. Let us suppose the theorem true for $n-1$ and let be $\mathfrak{M}_1 = \mathfrak{Q}_{n-1} + \mathfrak{N}_{n-1}$ the decomposition, where $\mathfrak{Q}_{n-1}$ is annihilated by the A_j and $\mathfrak{N}_{n-1} A_j \subseteq \mathfrak{N}_{n-1}$, $(j = 1, 2, \cdots, n-1)$. If $\mathfrak{L}$ is the kernel of the homomorphism $\mathfrak{Q}_{n-1} \sim \mathfrak{Q}_{n-1} A_n$, we have $\mathfrak{Q}_{n-1} = \mathfrak{L} + \mathfrak{T}$. As $\mathfrak{L} A_n = (0)$, $\mathfrak{T} A_n = \mathfrak{Q}_{n-1} A_n$, we have $\mathfrak{M}_1 = \mathfrak{L} + \mathfrak{T} + \mathfrak{N}_{n-1}$, with $\mathfrak{L} A_j = (0)$, $(j = 1, 2, \cdots, n)$, and $\mathfrak{M}_1 A_j = (\mathfrak{T} + \mathfrak{N}_{n-1}) A_j$. The submodule $\mathfrak{P} = ((\mathfrak{T} + \mathfrak{N}_{n-1}), (\mathfrak{T} + \mathfrak{N}_{n-1}) A_1, \cdots, (\mathfrak{T} + \mathfrak{N}_{n-1}) A_n)$ satisfies to $\mathfrak{P} A_j \subseteq \mathfrak{P}$, because, for instance, $(\mathfrak{T} + \mathfrak{N}_{n-1}) A_n A_j = \mathfrak{M}_1 A_n A_j \subseteq \mathfrak{M}_1 A_j = (\mathfrak{T} + \mathfrak{N}_{n-1}) A_j$. As $\mathfrak{P} \subseteq \mathfrak{M}_1$ and $\mathfrak{P} \supseteq \mathfrak{T} + \mathfrak{N}_{n-1}$, we have $\mathfrak{P} = \mathfrak{P} \cap \mathfrak{L} + \mathfrak{T} + \mathfrak{N}_{n-1}$ and $\mathfrak{L} = \mathfrak{P} \cap \mathfrak{L} + \mathfrak{O}$,

$\mathfrak{M}_1 = \mathfrak{P} \cap \mathfrak{L} + \mathfrak{O} + \mathfrak{T} + \mathfrak{N}_{n-1} = \mathfrak{P} + \mathfrak{O}$. The theorem is then proved taking $\mathfrak{Q}_1 = \mathfrak{O}$, $\mathfrak{N}_1 = \mathfrak{P}$, because $\mathfrak{O} \subseteq \mathfrak{L}$ gives $\mathfrak{O} A_j = (0)$

Corollary 11: *Let be $A_1, \cdots, A_n$ $\mathfrak{S}$-endomorphisms of the semi-simple module $\mathfrak{M}$. We have $\mathfrak{M} = \mathfrak{Q} + \mathfrak{N}$, where $\mathfrak{Q}$ is annihilated by the A_j and $\mathfrak{N} A_j \subseteq \mathfrak{N}$.*

An important example of semi-simple $\mathfrak{S}$-module is every module $\mathfrak{M}$ with a noetherian semi-simple ring of operators $\mathfrak{S}$, where $1 \in \mathfrak{S}$ acts as the identity endomorphism of $\mathfrak{S}$. The decomposition $\mathfrak{S} = \mathfrak{r}_1 + \cdots + \mathfrak{r}_n$, where the $\mathfrak{r}_j$ are simple right ideals, gives $m = m \cdot 1 = me_1 + \cdots + me_n$, where $e_j \in \mathfrak{r}_j$, $e_j^2 = e_j$, $e_j e_k = 0$, $(j \neq k)$. The correspondence $\mathfrak{r}_j \to m\mathfrak{r}_j$ is the null homomorphism or an isomorphism, and, in the decomposition of m the non-null summands belong to the simple submodules of the form $m\mathfrak{r}_j$. $\mathfrak{M}$ is then generated by its simple submodules and is a semi-simple $\mathfrak{S}$-module. When 1 does not act as the identity endomorphism, the decomposition $\mathfrak{M} = \mathfrak{N} + \mathfrak{N}'$ referred above is also true for every submodule $\mathfrak{N}$ where 1 acts as the identity endomorphism as we see from the decomposition of $\mathfrak{M}$ in two submodules $\mathfrak{M}'$ and $\mathfrak{M}''$, where $\mathfrak{M}'$ has 1 as identity endomorphism and $\mathfrak{M}''$ is the submodule whose elements are annihilated by $\mathfrak{S}$. Then, as $\mathfrak{N} \subseteq \mathfrak{M}'$, $\mathfrak{M}' = \mathfrak{N} + \mathfrak{L}$, we have $\mathfrak{M} = \mathfrak{N} + (\mathfrak{L} + \mathfrak{M}'') = \mathfrak{N} + \mathfrak{N}'$. It follows:

Theorem 34: *Let $\mathfrak{S}$ be a noetherian semisimple ring and $\mathfrak{M}$ an $\mathfrak{S}$-module. Every submodule $\mathfrak{N}$ where 1 acts as the identity endomorphism is a summand of a direct decomposition $\mathfrak{M} = \mathfrak{N} + \mathfrak{N}'$.*

As we have seen, a necessary condition for $\mathfrak{S}$ to be noetherian semi-simple is that every $\mathfrak{S}$-module may be written as a direct sum of the submodule annihilated by $\mathfrak{S}$ and a semi-simple $\mathfrak{S}$-submodule. O. Goldmann, [9], proved the converse proposition. The essential part of this proof is to show the existence of the unity. Then the use of the hypothesis in the ring $\mathfrak{S}$ gives the desired result, (cfr. [34, § 6]).

To the semi-simple modules with respect to a noetherian semi-simple ring of endomorphisms $\mathfrak{S}$, containing the identity, we can apply theorem 33, under a different feature, for which we need

Lemma 2: *Let $\mathfrak{M}$ be a module in whose absolute there exists a noetherian semi-simple ring of endomorphisms $\mathfrak{S}$ containing the identity. If $A \in \mathfrak{S}$ is such that $\mathfrak{M}A$ is generated by a finite number of simple $\mathfrak{S}$-submodules, every $\mathfrak{S}$-submodule $\mathfrak{M}_1$ can be written as $\mathfrak{M}_1 = \mathfrak{Q}_1 + \mathfrak{N}_1$, where $\mathfrak{Q}_1$ is annihilated by A and $\mathfrak{N}_1$ is also an $\mathfrak{S}$-submodule generated by a finite number of simple submodules.*

With the notations of theorem 33 (with $A_1 = A$), as $\mathfrak{M}_1 A = \mathfrak{N}_1'' A$ is generated by a finite number of simple submodules, we can find $t_1, t_2, \cdots, t_n \in \mathfrak{N}_1'' A$ such that $\mathfrak{M}_1 A = \mathfrak{N}_1'' A = t_1\mathfrak{S} + \cdots + t_n\mathfrak{S}$, and also $y_1, \cdots, y_n \in \mathfrak{N}_1''$ such that $y_j A = t_j$, $(j = 1, 2, \cdots, n)$. For every $x \in \mathfrak{M}_1$, we have $xA = yA = \Sigma t_i \sigma_i$, $(y \in \mathfrak{N}_1'', \sigma_i \in \mathfrak{S})$, and, then, $xA = yA = \Sigma (y_i A)\sigma_i = (\Sigma y_i \sigma_i) A$. Then the difference $z = x - \Sigma y_i \sigma_i$ belongs to $\mathfrak{N}_1'$ and $x = z + \Sigma y_i \sigma_i$ is the decomposition of x according to the one of $\mathfrak{M}_1 = \mathfrak{N}_1' + \mathfrak{N}_1''$. $\mathfrak{N}_1'' = y_1\mathfrak{S} + \cdots + y_n\mathfrak{S}$ is, consequently, finitely generated. The lemma has been proved, by setting $\mathfrak{Q}_1 = \mathfrak{N}_1'$, $\mathfrak{N}_1 = \mathfrak{N}_1''$.

As in theorem 33, we can continue and obtain $\mathfrak{P}_1 = (\mathfrak{N}_1'', \mathfrak{N}_1'' A)$ and $\mathfrak{O}_1$, if we suppose that $\mathfrak{M}_1 A \subseteq \mathfrak{M}_1$. This condition is verified if we apply this lemma to the module $\mathfrak{M}$.

Let $A_1, \cdots, A_n$ be $\mathfrak{S}$-endomorphisms of $\mathfrak{M}$ such that each $\mathfrak{M}A_i$ is generated by a finite number of simple submodules and let us suppose also $\mathfrak{M}_1 A_i \subseteq \mathfrak{M}_1$, $(i = 1, 2, \cdots, n)$. If we have obtained the decomposition $\mathfrak{M}_1 = \mathfrak{Q}_{n-1} + \mathfrak{N}_{n-1}$, as in theorem 33, we can write $\mathfrak{Q}_{n-1} = \mathfrak{L} + \mathfrak{T}$, where the submodule $\mathfrak{T}$ has a finite basis. As $\mathfrak{N}_{n-1}$ has a finite basis, $\mathfrak{P}$ has also a finite one. The decomposition $\mathfrak{M}_1 = \mathfrak{P} + \mathfrak{O} = \mathfrak{Q}_1 + \mathfrak{N}_1$ verifies the

Theorem 35: *Let $\mathfrak{M}$ be a module in whose absolute there exists a noetherian semi-simple ring of endomorphisms $\mathfrak{S}$ containing the identity. If $A_1, \cdots, A_n \in \mathfrak{S}$ are transformations such that every $\mathfrak{M}A_i$ are generated by a finite number of simple $\mathfrak{S}$-submodules and such that for the $\mathfrak{S}$-submodule $\mathfrak{M}_1$ we have $\mathfrak{M}_1 A_i \subseteq \mathfrak{M}_1$, $(i = 1, 2, \cdots, n)$, we have the decomposition $\mathfrak{M}_1 = \mathfrak{Q}_1 + \mathfrak{N}_1$, where $\mathfrak{Q}_1$ is annihilated by the A_i, $\mathfrak{N}_1$ has a finite basis with respect to $\mathfrak{S}$ and $\mathfrak{N}_1 A_i \subseteq \mathfrak{N}_1$.*

Let us suppose now that $\mathfrak{R}$ is a noetherian simple ring of operators of $\mathfrak{M}$. If $\mathfrak{M}$ is a non-trivial $\mathfrak{R}$-module $(\mathfrak{M}\mathfrak{R} \neq (0))$, we can represent faithfully $\mathfrak{R}$ in the ring of endomorphisms of $\mathfrak{M}$, without making the hypothesis that $1 \in \mathfrak{R}$ is unitary operator. But, if $1 \in \mathfrak{R}$ acts as the identity, we have $\mathfrak{M} = \Sigma \mathfrak{m}_\mu$, $(\mu \in M)$, where the $\mathfrak{m}_\mu$ are $\mathfrak{R}$-isomorphic, as they are $\mathfrak{R}$-isomorphic of a simple right ideal of $\mathfrak{R}$. We can then apply to $\mathfrak{M}$ the following extension of the II Theorem of Wedderburn-Artin:

Theorem 36: *The ring of $\mathfrak{R}$-endomorphisms of a module $\mathfrak{M}$ over a noetherian simple ring $\mathfrak{R}$, whose identity acts as the identity endomorphism, is a complete ring of row-summable transfinite matrices over the division ring isomorphic to the ring of $\mathfrak{R}$-endomorphisms of a simple right ideal of $\mathfrak{R}$.*

We can now give the structure theorem for the ring of $\mathfrak{R}$-endomorphisms of a module $\mathfrak{M}$ over a noetherian semi-simple ring $\mathfrak{R}$. Let be $\mathfrak{R} = \mathfrak{A}_1 + \cdots + \mathfrak{A}_t$ the decomposition of $\mathfrak{R}$ in simple rings, under the hypothesis that $1 \in \mathfrak{R}$ acts as the identity. We have then $\mathfrak{M} = \mathfrak{M}\mathfrak{R} = \mathfrak{M}\mathfrak{A}_1 + \cdots + \mathfrak{M}\mathfrak{A}_t$. Let be $F_i, (i = 1, 2, \cdots, t)$, the projections of $\mathfrak{M}$ over the $\mathfrak{M}_i = \mathfrak{M}\mathfrak{A}_i$. We know that $\overline{\mathfrak{R}} = \Sigma \overline{\mathfrak{R}}_{ij}$, $(i, j = 1, 2, \cdots, t)$, with $\overline{\mathfrak{R}}_{ij} = F_i \overline{\mathfrak{R}} F_j$. As every $\mathfrak{M}_i$ is $\mathfrak{R}$-semi-simple and may be expressed as a direct discrete sum of isomorphic modules, not isomorphic of the summands of $\mathfrak{M}_j, j \neq i$, we will show that $\overline{\mathfrak{R}}_{ij} = (0)$. The homomorphism $\mathfrak{M}_i \sim \mathfrak{M}'_j \subset \mathfrak{M}_j$ carries every simple submodule of $\mathfrak{M}$ in (0). It carries, then, $\mathfrak{M}_i$ in (0) and $\overline{\mathfrak{R}}_{ij}$ has only the zero element. By that, we conclude that $\overline{\mathfrak{R}} = \Sigma \overline{\mathfrak{R}}_{ii}, (i = 1, 2, \cdots, t)$. As the structure of the $\overline{\mathfrak{R}}_{ii}$ is given by the earlier theorem, we can give the following proposition, which may be considered as the extension of the I Theorem of Wedderburn-Artin:

Theorem 37: *The ring of $\mathfrak{R}$-endomorphisms of a module $\mathfrak{M}$ over a noetherian semi-simple ring $\mathfrak{R}$, whose identity acts as the identity endomorphism, is isomorphic to a direct sum of a finite number of complete matrix rings, like those of the later theorem. The number of summands of $\overline{\mathfrak{R}}$ is the number of simple rings of the decomposition of $\mathfrak{R}$ or the number of systems of not isomorphic simple right ideals.*

In correlation with the propositions already given, we can give the following one [29]:

Theorem 38: *Let $\mathfrak{R}$ be a ring with identity, which is also the identity of $\mathfrak{S}$, a noetherian simple subring of $\mathfrak{R}$; then $\mathfrak{R}$ can be written $\mathfrak{R} = \Sigma w_\nu \mathfrak{S}$, $(\nu \in N)$, as a direct discrete sum of the $\mathfrak{S}$-submodules $w_\nu \mathfrak{S}$, all $\mathfrak{S}$-isomorphic to $\mathfrak{S}$, and the order $(\mathfrak{R}/\mathfrak{S})$ is the cardinality of N.* As $\mathfrak{R}$ is a module in whose absolute the ring $\mathfrak{R}$, and, consequently, $\mathfrak{S}$ are faithfully represented, we have $\mathfrak{R} = \Sigma \mathfrak{m}_\nu$, where the $\mathfrak{m}_\nu$ are $\mathfrak{S}$-simple. For every $0 \neq w_\nu \in \mathfrak{m}_\nu$, we have $\mathfrak{m}_\nu = w_\nu \mathfrak{S}$, with the w_ν $\mathfrak{S}$-independent.

We can also obtain a stucture theorem for $\overline{\mathfrak{R}}$, by applying to $\mathfrak{R} = \Sigma w_\nu \mathfrak{S}$ a proposition given in § 8 and using the fact that, as $\mathfrak{R}$ has identity, $\overline{\mathfrak{R}} = \mathfrak{R}_r$ is anti-isomorphic to $\mathfrak{R}$.

The following proposition is an useful one, specially when used in some particular condition. Let $\mathfrak{M}$ be a module and $\mathfrak{S}$ a noetherian semi-simple ring of endomorphisms, containing the identity. Let $\mathfrak{T}$ be a subset of $\overline{\mathfrak{S}}$ that generates in $\overline{\mathfrak{S}}$ a nilpotent subring $\overline{\mathfrak{T}}_d$, whose exponent is $\sigma, (\overline{\mathfrak{T}}_d = (0), \overline{\mathfrak{T}}_d^{\sigma-1} \neq (0))$. For the $\mathfrak{S}$-submodule $\mathfrak{M}_\sigma = \mathfrak{M}\overline{\mathfrak{T}}_d^{\sigma-1} \neq (0)$, we have $\mathfrak{M}_0 \overline{\mathfrak{T}}_d = (0)$ and $\mathfrak{M} = \mathfrak{M}_0 + \mathfrak{M}'_0$, as $\mathfrak{M}$ is $\mathfrak{S}$-semi-simple. For every $x \in \mathfrak{M}$, we have $x = x_0 + x'_0 = x\alpha + x\varepsilon$, $(x_0 \in \mathfrak{M}_0, x'_0 \in \mathfrak{M}'_0; \alpha, \varepsilon \in \overline{\mathfrak{S}})$. If $\gamma \in \overline{\mathfrak{T}}_d$, it is $x\gamma = x\varepsilon\gamma$, and, consequently, $\varepsilon\gamma = \gamma$. We have also $\mathfrak{M}_0 \varepsilon = (0)$. Then:

Theorem 39: *If $\mathfrak{S}$ is a noetherian semi-simple ring of endomorphisms of a module $\mathfrak{M}$, containing the identity, and if $\overline{\mathfrak{T}} \subseteq \overline{\mathfrak{S}}$ generates a nilpotent ring $\overline{\mathfrak{T}}_d$ in the absolute of $\mathfrak{M}$, there exists an idempotent $\varepsilon \in \overline{\mathfrak{S}}$ such that $\varepsilon\gamma = \gamma$, for every $\gamma \in \overline{\mathfrak{T}}$ (or $\overline{\mathfrak{T}}_d$), and the endomorphism ε is not an automorphism.*

A semi-simple module is sudirectly irreducible if, and only if, it is simple [30, pgs, 134 and ff.]. As we have referred in § 1 we will correct considerations of [30] in correlation with the theory of subdirectly irreducible modules. In [30, pg. 139], we have given the following

Theorem 40: *Let $\mathfrak{M}$ be an $\mathfrak{S}$-module and $\mathfrak{S}$ be a comutative ring. It is a sufficient condition for the subdirect* *irreducibility of* $\mathfrak{M}$ *that the following conditions be realised:* 1) *there exists an* $\mathfrak{S}$*-submodule* $\mathfrak{L}=|mx_0+x_0\mathfrak{S}|\neq(0)$ *whose annihilator is* $\Delta\neq\mathfrak{S}$; 2) $\mathfrak{S}/\Delta$ *is a field whose identity is the identity operator of* $\mathfrak{L}$; 3) $\mathfrak{L}$ *and* Δ *are reciprocal annihilators;* 4) *for each* $x\notin\mathfrak{L}$, *there exists* $D_1\in\Delta$ *such that* $xD_1=x_0$. Here we will correct the note which follows the proof of this theorem in [30]. It is not necessary to exclude the hypothesis $\Delta=(0)$. Actually, by 3), we have $\mathfrak{L}=\mathfrak{M}$, which excludes 4. After, we fall in the proposition: if $\mathfrak{M}$ is a module over the field $\mathfrak{S}$, whose identity is the identity operator, and if $\mathfrak{M}$ can be generated by only one element, then $\mathfrak{M}$ is $\mathfrak{S}$-simple. By the hypotesis $\Delta=(0)$, with the exclusion of 3) and 4), it follows that $\mathfrak{L}$ is $\mathfrak{S}$-irreducible, but we have not proved the subdirect irreducibility of $\mathfrak{M}$. On the contrary, the decomposition $\mathfrak{M}=\mathfrak{M}'+\mathfrak{M}''$, as in theorem 34, shows that $\mathfrak{L}\subseteq\mathfrak{M}'$ is a direct summand of the semi-simple module $\mathfrak{M}'$ and, consequently, a direct summand of $\mathfrak{M}$.

10) **Some questions on irreducible rings** — In the questions which we will treat, we will use the propositions of the former §§.

When $\mathfrak{M}$ is a module over the division ring $\mathfrak{D}$ whose identity acts as the unitary operator, as $\mathfrak{D}$ is a noetherian simple ring, we can write $\mathfrak{M}=\Sigma u_\mu\mathfrak{D}$, $(\mu\in M)$, where the $u_\mu\in\mathfrak{M}$ and the submodules $u_\mu\mathfrak{D}$ are $\mathfrak{D}$-simple and, consequently, isomorphic to $\mathfrak{D}$.

Let us consider now that $\mathfrak{M}$, whose absolute is $\mathfrak{A}$, is $\mathfrak{A}$-irreducible. Obviously, $\mathfrak{A}$ is irreducible, and we have the following

Theorem 41: $\mathfrak{M}$ *is irreducible with respect to the absolute* $\mathfrak{A}$, *if, and only if,* $\mathfrak{A}$ *contains a prime field containing the identity,* [9].

The necessity: As $\mathfrak{M}$ is $\mathfrak{A}$-irreducible, its commutator $\overline{\mathfrak{A}}$ (the center of $\mathfrak{A}$) is a field that contains a prime field. The sufficiency: As there exists a prime field $\mathfrak{P}\subseteq\mathfrak{A}$, we may consider $\mathfrak{M}$ as a $\mathfrak{P}$-module and we can wite $\mathfrak{M}=\Sigma u_\mu\mathfrak{P}$, $(\mu\in M)$, as direct discrete sum of simple and isomorphic $\mathfrak{P}$-submodules [cfr. 30, §§ 8 and 9]. As the commutator of $\mathfrak{P}$, in the endomorphism ring of $u_\mu\mathfrak{P}$, is $\mathfrak{P}$, and as there are not in the $u_\mu\mathfrak{P}$ proper $\mathfrak{P}$-submodules, except (0), theorem 32 shows that $\mathfrak{M}$ is already $\mathfrak{P}$-irreducible.

Once proved the proposition we can give to the sufficiency condition a more general formulation: Let $\mathfrak{A}$ be the absolute of $\mathfrak{M}$ and let us consider $\mathfrak{B}=\overline{\mathfrak{K}}\subseteq\mathfrak{A}$ the commutator, in $\mathfrak{A}$, of a field $\mathfrak{P}\subseteq\overline{\mathfrak{K}}\subseteq\mathfrak{A}$ containing the identity. Then $\mathfrak{M}$ is $\mathfrak{B}$-irreducible and the ring $\mathfrak{B}$ is also irreducible.

Let us suppose now that the division ring $\mathfrak{D}$, of endomorphisms of $\mathfrak{M}$, contains the identity. As $\mathfrak{M}=\Sigma u_\mu\mathfrak{D}$, $(\mu\in M)$, if $\mathfrak{Q}$ is the center of $\mathfrak{D}$, then $\mathfrak{M}$ is $\mathfrak{A}$-irreducible and $\overline{\mathfrak{Q}}$-irreducible. As formely, we can prove that $\mathfrak{M}$ is $\overline{\mathfrak{D}}$-irreducible. In fact, the non-null $\mathfrak{D}$-endomorphisms of $u_\mu\mathfrak{D}$ are automorphisms constituting (with the null-endomorphism) a division ring $\mathfrak{D}'$, anti-isomorphic to $\mathfrak{D}$. We can define a $\sigma\in\mathfrak{D}'$ by the correspondence $u_\mu\to u_\mu\sigma=u_\mu d$. And we can write $u_\mu\mathfrak{D}=u_\mu\mathfrak{D}'$ where $d\in\mathfrak{D}$ and $\sigma\in\mathfrak{D}'$ are in the correspondence defined above. Then $u_\mu\mathfrak{D}$ has not $\mathfrak{D}'$-submodules, except, the trivial ones, and $\mathfrak{M}$ has not $\overline{\mathfrak{D}}$-submodules, except, also, the trivial ones. In a more direct way, let be $\mathfrak{m}\neq(0)$ a $\overline{\mathfrak{D}}$-submodule of $\mathfrak{M}$ and let $x=u_\alpha d_\alpha+\cdots+u_\xi d_\xi\neq 0$ be an element of $\mathfrak{M}$. The $\overline{\mathfrak{D}}$-endomorphism, of $\mathfrak{M}$, which applies $u_\alpha d_\alpha$ in $u_\lambda d_\lambda$ and u_μ, $(\mu\neq\alpha)$, in 0, applies also x in $u_\lambda d_\lambda$. Then, for every fixed λ, we have $u_\lambda\mathfrak{D}\subseteq\mathfrak{m}$, and consequently, $\mathfrak{m}=\mathfrak{M}$.

Let us suppose now that the absolute of $\mathfrak{M}$ has a noetherian simple ring $\mathfrak{S}$ of endomorphisms, where $1\in\mathfrak{S}$

is the identical endomorphism. By theorem 41, $\mathfrak{M}$ is $\mathfrak{A}$-irreducible, and, by the later §, we can write $\mathfrak{M}=\Sigma\,\mathfrak{m}_\mu$, $(\mu \in M)$, where each $\mathfrak{m}_\mu$ is $\mathfrak{S}$-isomorphic of any minimal right ideal of $\mathfrak{S}$. Evidently, $\mathfrak{M}$ is not $\overline{\mathfrak{S}}$-irreducible ($\overline{\mathfrak{S}}$ is not a division ring), and, consequently, if there exists $\overline{\mathfrak{S}}$-submodules of $\mathfrak{M}$, there exists also $\wedge$-submodules of a minimal right ideal $e\mathfrak{S}$, of $\mathfrak{S}$, where $\wedge$ is the division ring of the $\mathfrak{S}$-endomorphisms of $e\mathfrak{S}$. We have:

Theorem 42: *Let be $\mathfrak{A}$ the absolute of $\mathfrak{M}$. If $\overline{\mathfrak{S}}\subseteq\mathfrak{A}$ is the commutator, in $\mathfrak{A}$, of a simple noetherian ring $\mathfrak{S}$ with the identity of $\mathfrak{A}$, there is a $1-1$ correspondence between the $\overline{\mathfrak{S}}$-submodules of $\mathfrak{M}$ and the $e\mathfrak{S}e$-submodules of a minimal right ideal $e\mathfrak{S}$, of $\mathfrak{S}$.* The elements of $e\mathfrak{S}e$ may be considered as left operators of $e\mathfrak{S}$. We will show that $\mathfrak{S}$ and $\overline{\mathfrak{S}}$ are reciprocal commutators.

It could be thought as a case different from the later the case where we could find in $\mathfrak{A}$ a simple ring $\mathfrak{S}$ with minimal right ideals and identity, whose identity is also the identity of $\mathfrak{A}$. The considerations of [26, § 4] show that this case is included in the later as $\mathfrak{S}$ is a simple noetherian ring, [4], [29].

We have, in this §, noted that every module $\mathfrak{M}$ over a division ring $\mathfrak{D}$ is semi-simple. If $\mathfrak{N}$ is a $\mathfrak{D}$-submodule, there exists a submodule $\mathfrak{N}'$ such that $\mathfrak{M}=\mathfrak{N}+\mathfrak{N}'$. There is always an idempotent E such that $\mathfrak{M}E=\mathfrak{N}$ and for every $x\in\mathfrak{N}'$ we have $x'E=0$. If $\mathfrak{N}$ is finite over $\mathfrak{D}$, let be $u_\alpha, u_\beta, \cdots, u_\lambda$ an independent basis for $\mathfrak{N}$, with $\mathfrak{M}= =\Sigma\, u_\mu\mathfrak{D}$, $(\mu\in M)$. Then $E=E_\alpha+E_\beta+\cdots+E_\lambda$ is an idempotent of $\overline{\mathfrak{D}}$ such that $\mathfrak{M}E=\mathfrak{N}=\mathfrak{M}E_\alpha+\mathfrak{M}E_\beta+ +\cdots+\mathfrak{M}E_\lambda$, with $\mathfrak{M}E_\alpha=u_\alpha\mathfrak{D},\cdots$ as the idempotents $E_\alpha, E_\beta, \cdots, E_\lambda$ are ortogonal. If $\mathfrak{M}$ is infinite over $\mathfrak{D}$, the finite linear transformations belonging to $\overline{\mathfrak{D}}$ constitute a

two-sided ideal $\overline{\mathfrak{F}}$, of $\mathfrak{A}$. This ideal is irreducible in $\mathfrak{M}$ over $\mathfrak{D}$. We may, then, consider the irreducible rings $\overline{\mathfrak{S}}$, subrings of $\overline{\mathfrak{D}}$, different from $\overline{\mathfrak{D}}$, as $1\notin\overline{\mathfrak{F}}$.

To continue the study of modules over division rings, we will prove the following simple property: if $\mathfrak{m}=v\mathfrak{D}$, the division ring $\mathfrak{D}$ is closed in the absolute $\mathfrak{a}$ of $\mathfrak{m}$. In fact, as $\mathfrak{D}$ and $v\mathfrak{D}$ are $\mathfrak{D}$-isomorphic, their absolutes are isomorphic, and we may suppose that in the latter isomorphism the elements of $\mathfrak{D}$ are in correspondence. As $\mathfrak{D}$ is a ring with identity, the right multiplications of $\mathfrak{D}$ constitute the ring $\mathfrak{D}$ of endomorphisms which is the reciprocal commutator of the ring $\mathfrak{D}'$ of the left multiplications of $\mathfrak{D}$. We have $\mathfrak{D}'=\overline{\mathfrak{D}}$ and $\overline{\mathfrak{D}'}=\overline{\overline{\mathfrak{D}}}=\mathfrak{D}$. And the same can be said of $\mathfrak{D}$, as the ring of endomorphisms of $v\mathfrak{D}$.

Let us return, now, to the general question of a module $\mathfrak{M}=\Sigma\, u_\mu\mathfrak{D}$ over the division ring $\mathfrak{D}$. Corollary 10, of theorem 31, shows that we have $\overline{\overline{\mathfrak{D}}}=\mathfrak{D}$, that is, $\mathfrak{D}$ is closed in the absolute $\mathfrak{A}$ of $\mathfrak{M}$. A direct proof of this can be given. If $\Theta\in\overline{\overline{\mathfrak{D}}}$, we know, from theorem 31, that Θ induces a $\overline{\mathfrak{B}}_\mu$-endomorphism in $u_\mu\mathfrak{D}$, which we will denote by θ_μ. We have proved that $\theta_\mu=d_\mu\in\mathfrak{D}$. If we can show that d_μ is independent of μ, the desired proof remains established. With the notations of the theory of direct discrete sums of isomorphic modules, we have $u_\mu\Delta_{\mu\nu}=u_\nu$ and $u_\nu\Theta=(u_\mu E_\mu\Delta_{\mu\nu})\Theta=(u_\mu E_{\mu\nu})\Theta=(u_\mu\Theta)E_{\mu\nu}=(u_\mu\theta_\mu)E_{\mu\nu}= =(u_\mu d_\mu)E_{\mu\nu}=(u_\mu E_{\mu\nu})d_\mu=u_\nu d_\mu$. Then $\theta_\nu=d_\mu$, as we desire.

A more direct method is given by Artin-Whapples-Jacobson. Clearly, given $\mathfrak{D}$ and $\overline{\mathfrak{D}}$, for every system of elements $x_1,\cdots,x_r\in\mathfrak{M}$, $\mathfrak{D}$-independent, there exists always an $A \in \bar{\mathfrak{D}}$ which applies the x_i in $y_i, (i = 1, 2, \cdots, t)$, $\mathfrak{D}$-independent or not, (cfr. the notion of *dense ring* given in [4]). Let us take $\Theta \in \bar{\bar{\mathfrak{D}}}$ and let be $x \to x\Theta = y$. For every x, x and y are not $\mathfrak{D}$-independent. Because if x and y could be independent, for some $A \in \bar{\mathfrak{D}}$ we could have $xA = x$, $yA = 0$, $x \to x\Theta = y$, $x = xA \to (xA)\Theta = (x\Theta)A = yA = 0$, which is an absurd. Then we have $x \to x\Theta = y = xd$, $(d \in \mathfrak{D})$. For every $z \in \mathfrak{M}$, we show that we also have $z \to z\Theta = zd$. Let $B \in \bar{\mathfrak{D}}$ be such that $xB = z$. Then $x \to x\Theta = xd$, $z = xB \to (xB)\Theta = (x\Theta)B = (xd)B = (xB)d = zd$. We have the following

THEOREM 43: *Let $\mathfrak{A}$ be the absolute of $\mathfrak{M}$. To every division ring contained in $\mathfrak{A}$, with the same identity of $\mathfrak{A}$, correspond different commutators, which are irreducible rings. And if $\mathfrak{M}$ is $\mathfrak{A}$-irreducible, its center is a prime field.*

If $\mathfrak{R}$ is an irreducible ring and $\mathfrak{D}$ its commutator, $\mathfrak{Z} = \mathfrak{R} \cap \mathfrak{D}$ is the center of $\mathfrak{R}$. If $\mathfrak{R}$ is closed, the common center of $\mathfrak{R}$ and $\mathfrak{D}$ is the field $\mathfrak{Z}$. But, generally, $\mathfrak{R} \cap \mathfrak{D} \subseteq \bar{\mathfrak{D}} \cap \mathfrak{D}$. $\mathfrak{A}$ is always a closed ring. If $\mathfrak{A}$ is irreducible, we have a special case of

THEOREM 44: *If $\mathfrak{R} = \bar{\mathfrak{D}}$ is closed irreducible ring, with $\mathfrak{D}$ as its commutator, its center is the field $\bar{\mathfrak{D}} \cap \mathfrak{D}$.*

When $\mathfrak{E}$ is an irreducible ring of endomorphisms of a module $\mathfrak{M}$, every two-sided ideal $\mathfrak{a} \neq (0)$, of $\mathfrak{E}$, is also irreducible. In fact, according ARTIN-WHAPPLES-JACOBSON, we will show that, for every $0 \neq x \in \mathfrak{M}$, there exists an $H \in \mathfrak{a}$ such that $xH = y$, for every $y \in \mathfrak{M}$. Let be $0 \neq A \in \mathfrak{a}$. There exists $0 \neq t \in \mathfrak{M}$ such that $z = tA \neq 0$. If $B \in \mathfrak{E}$ is such that $xB = t$, we have $xBA = tA = z$; if $C \in \mathfrak{E}$ is such that $zC = y$, we have $zBAC = zC = y$, and $BAC \in \mathfrak{a}$. To

give a proposition like the one of [29, theorem 4, pg. 93] and of [9, theorem 3, pg. 951], we will prove that every ring $\mathfrak{R}$, closed and irreducible is an irreducible ideal ring, [4]. Let us suppose $\mathfrak{R}$ faithfully represented in the absolute $\mathfrak{A}$ of $\mathfrak{M}$ and let be $\bar{\mathfrak{D}} = \mathfrak{R}$. By hypothesis, $\mathfrak{R}$ is the set of all $\mathfrak{D}$-endomorphisms of $\mathfrak{M}$. If $0 \neq x \in \mathfrak{M}$ and we write $\mathfrak{M} = x\mathfrak{D} + \mathfrak{N}'$, we will prove that the ideal $\mathfrak{r}' \subseteq \mathfrak{R}$ which annihilates $\mathfrak{N}'$ is a minimal right ideal. If $0 \neq B \in \mathfrak{r}'$ and $C \in \mathfrak{r}'$ is any element, we cannot have $xB = 0$, because, in that case, B will annihilate $\mathfrak{M}$, and, consequently, $B = 0$. Then, as $xB \neq 0$, there exists $D \in \mathfrak{R}$ such that $xBD = xC$, or $x(BD - C) = 0$, and $BD - C = 0$, as $BD - C \in \mathfrak{r}'$. The equality $C = BD$ shows that $\mathfrak{r}'$ is a minimal right ideal.

The proposition already referred is the following one, which involves the considerations of this § and of § 3:

THEOREM 45: *Let $\mathfrak{R}$ be a closed irreducible ring. Then there exists a division ring $\mathfrak{D}$ and a linear space $\mathfrak{M}$, over $\mathfrak{D}$, in which $1 \in \mathfrak{D}$ acts as the unitary operator, such that $\mathfrak{R}$ is the set of all $\mathfrak{D}$-linear transformations of $\mathfrak{M}$; $\mathfrak{R}$ can also be faithfully represented as the ring of all $\mathfrak{D}'$-linear transformations of a minimal right ideal $\mathfrak{r}'$ where $\mathfrak{D}'$ is the division ring of $\mathfrak{R}$-endomorphisms of $\mathfrak{r}'$. Conversely, the set of $\mathfrak{D}$-linear transformations of a linear space $\mathfrak{M}$, over the division ring $\mathfrak{D}$, where $1 \in \mathfrak{D}$ acts as the unitary operator, is a closed irreducible ring $\mathfrak{R}$; the linear space $\mathfrak{M}$ is $\mathfrak{R}$-irreducible, and, consequently, is $\mathfrak{R}$-isomorphic to every minimal right ideal of $\mathfrak{R}$, and the division ring $\mathfrak{D}$ is the set of $\mathfrak{R}$-endomorphisms of the linear space. From $\mathfrak{R}$, $\mathfrak{D}$ and $\mathfrak{M}$ are well defined, apart isomorphisms.*

We have seen that there exists irreducible rings which are not closed. In the following considerations,

$\mathfrak{B} \neq (0)$ will always be an arbitrary irreducible ring of endomorphisms of a module $\mathfrak{M}$, such that $\overline{\mathfrak{B}} = \mathfrak{D}$ (division ring) is its commutator. We will prove the following important proposition:

THEOREM 46: *For* $0 \neq x_1 \in \mathfrak{M}$, *with* $\mathfrak{B} \neq (0)$ *an irreducible ring of endomorphisms of* $\mathfrak{M}$ *having* $\mathfrak{D}$ *as its commutator, the right ideal* $\overline{\mathfrak{J}}_1$, *of* $\mathfrak{B}$, *which annihilates* $[x_1] = x_1 \mathfrak{D}$, *is* $\neq 0$ *as for* $0 \neq x_0 \notin [x_1]$ *we have* $x_0 \overline{\mathfrak{J}}_1 \neq (0)$, (*cf.* [3]). Evidently, the case in which the order $(\mathfrak{M}/\mathfrak{D})$ is one, because, in that case, $x\mathfrak{B} = \mathfrak{M} = x\mathfrak{D}$, for every $0 \neq x \in \mathfrak{M}$, is such that only the right ideal (0) will annihilate $[x_1] = \mathfrak{M}$, and we may not find $x_0 \notin [x_1]$. In the other cases the proof is as follows. Clearly, the right ideal of the absolute which annihilates $[x_1]$ is $\neq 0$. But we are considering the annihilator contained in $\mathfrak{B}$. Let us take $0 \neq x_0 \notin [x_1]$ and suppose that $x_0 \overline{\mathfrak{J}}_1 = 0$. If $A_1 \in \mathfrak{B}$ is such that $x_1 A_1 = x_1$, the correspondence

$$x_1 A \rightarrow x_0 A_1 A, \qquad (\text{for every } A \in \mathfrak{B}), \tag{4}$$

is a $\mathfrak{B}$-endomorphism of $\mathfrak{M}$, as we will see. Firstly, it is $x_1 \mathfrak{B} = \mathfrak{M}$; after, if we have a C such that $x_1 A = x_1 C$, $(C \in \mathfrak{B})$, or $x_1(A-C) = x_1 A_1 (A-C) = 0$, is $A_1(A-C) \in \overline{\mathfrak{J}}_1$, and, consequently, $x_0 A_1 (A-C) = 0$, or $x_0 A_1 A = x_0 A_1 C$. Then, in the correspondence (4), $x_1 A$ and $x_1 C$ have the same correspondent. Let $\alpha_1 \in \mathfrak{D}$ represent the endomorphism (4). We have

$$x_1 A \rightarrow x_0 A_1 A = x_1 A \alpha_1, \qquad x_1 = x_1 A_1 \rightarrow x_0 A_1 A_1 = x_1 \alpha_1 .$$

As $x_1(A - A_1 A) = 0$, we have also $x_0(A - A_1 A) = (x_0 - x_0 A_1) A = 0$. As A is any element of $\mathfrak{B}$, we have

$x_0 - x_0 A_1 = 0$, or $x_0 = x_0 A_1$. Thus $x_0 = x_0 A_1 = x_0 A_1 A_1 = x_1 \alpha_1 \in [x_1]$, which is against the hypothesis. Then $x_0 \overline{\mathfrak{J}}_1 = (0)$ is an absurd and the theorem is proved. Consequently, we have: 1) $x_0 \overline{\mathfrak{J}}_1 = \mathfrak{M}$; 2) if $x_0, x_1 \in \mathfrak{M}$ are $\mathfrak{D}$-independent, there exists $B_0, B_1 \in \mathfrak{B}$ such that $x_0 B_0 = x_0$, $x_1 B_0 = 0$, $x_0 B_1 = 0$, $x_1 B_1 = x_1$, [4], [24]. This second consequence can be proved as follows: there exists $A_0 \in \overline{\mathfrak{J}}_1$ such that $x_0 A_0 = x_0$ and $A_1' \in \overline{\mathfrak{J}}_1$ such that $x_0 A_1' = x_0 A_1$. Taking then $B_0 = A_0$, $B_1 = A_1 - A_1'$, we have $x_0 B_0 = x_0$, $x_1 B_0 = 0$, $x_1 B_1 = x_1(A_1 - A_1') = x_1 A_1 = x_1$, $x_0 B_1 = x_0 \cdot (A_1 - A_1') = x_0 A_1 - x_0 A_1' = 0$. And we can now give the following more general proposition:

THEOREM 47: *If* $\mathfrak{B} \neq (0)$ *is an irreducible ring of endomorphisms of* $\mathfrak{M}$ *and it is* $\overline{\mathfrak{B}} = \mathfrak{D}$; *if* $x_1, \cdots, x_n \in \mathfrak{M}$ *are* $\mathfrak{D}$-*independent and* $A_1, \cdots, A_n \in \mathfrak{B}$ *such that* $x_i A_i = x_i$, $x_i A_j = 0$, $(i \neq j)$, *the ideal* $\overline{\mathfrak{J}}$, *of* $\mathfrak{B}$, *which annihilates the* $\mathfrak{D}$-*subspace of* $\mathfrak{M}$ *generated by* $x_1, \cdots, x_n$ *and represented by* $[x_1, \cdots, x_n]$ *is* $\neq (0)$, *and we have, for each* $x_0 \notin [x_1, \cdots, x_n]$, $x_0 \overline{\mathfrak{J}} \neq (0)$. As in theorem 46, the hypothesis $(\mathfrak{M}/\mathfrak{D}) > n$ carries that is $\neq (0)$ the right ideal of the absolute which annihilates $[x_1, \cdots, x_n]$. But we are searching the annihilator contained in $\mathfrak{B}$. For $0 \neq x_0 \notin [x_1, \cdots, x_n]$ and some i, we will study

$$x_i A \rightarrow x_0 A_i A, \qquad (\text{for every } A \in \mathfrak{B}).$$

We conclude that this correspondence is a $\mathfrak{B}$-endomorphism and we can write [we are supposing $x_0 \overline{\mathfrak{J}} = (0)$]:

$$x_i A \rightarrow x_0 A_i A = x_i A \alpha_i, \qquad (\alpha_i \in \mathfrak{D}).$$

Analogously, we have

$$x_i = x_i A_i \rightarrow x_i A_i \alpha_i = x_0 A_i A_i = x_i \alpha_i ,$$

and also $A_i^2 - A_i \in \overline{\mathfrak{J}}$, $x_0 A_i A_i = x_0 A_i$. Taking $D = \Sigma A_i$, we have

$$x_j(A - DA) = x_j A - x_j DA = 0,$$
$$x_0(A - DA) = (x_0 - x_0 D) A = 0,$$

and, as A is any element of $\mathfrak{S}$, we conclude

$$x_0 - x_0 D = 0, \qquad x_0 = x_0 D = x_0 \Sigma A_i.$$

The $\mathfrak{S}$-endomorphisms $\alpha_1, \cdots, \alpha_n \in \mathfrak{D}$ give then

$$x_0 = x_0 \Sigma A_i = x_0 \Sigma A_i A_i = \Sigma x_i \alpha_i \in [x_1, \cdots, x_n],$$

against the hypothesis that $x_0 \notin [x_1, \cdots, x_n]$. It is an absurd to suppose $x_0 \overline{\mathfrak{J}} = (0)$.

Consequently, if $(\mathfrak{M}/\mathfrak{D}) > n$, we can find, from the A_i, taking $x_{n+1} \notin [x_1, \cdots, x_n]$, a system of $B_1, \cdots, B_n, B_{n+1} \in \mathfrak{S}$ such that $x_i B_i = x_i$, $x_j B_i = 0$, $(i \neq j)$, $(i, j = 1, 2, \cdots \cdots, n, n+1)$, as we will see. Taking $x_{n+1} = x_0$, we know that $x_0 \overline{\mathfrak{J}} \neq (0)$, and, then, $x_0 \overline{\mathfrak{J}} = \mathfrak{M}$. There exists $A_0 \in \overline{\mathfrak{J}}$ such that $x_0 A_0 = x_0$. And there exists also $A_i' \in \overline{\mathfrak{J}}$, $(i = 1, 2, \cdots, n)$, for which we have $x_0 A_i' = x_0 A_i$. Taking then $A_0 = B_{n+1}$, $B_i = A_i - A_i'$, we see that

$$x_{n+1} B_{n+1} = x_{n+1}, \qquad x_i B_{n+1} = x_i A_0 = 0,$$
$$x_i B_i = x_i A_i - x_i A_i' = x_i A_i = x_i, \quad x_j B_i = x_j (A_i - A_i) = x_j A_i = 0,$$

$(j \neq i, j = 1, 2, \cdots, n)$, as we desire.

The relations between theorem 47 and the one of Chevalley-Jacobson, [24, § 6], can be explained easily. By theorem 47, we conclude that every irreducible ring of endomorphisms with $\mathfrak{D}$ as its commutator, is a dense ring in $\mathfrak{M}$ over $\mathfrak{D}$. Conversely, by the considerations

before theorem 43, we see that we can apply them to every dense ring of $\mathfrak{D}$-endomorphisms of a module $\mathfrak{M}$, and, consequently, this dense ring is an irreducible one and has $\mathfrak{D}$ as its commutator, [4].

As we have seen that given $[x_1, \cdots, x_n]$ its annihilator $\overline{\mathfrak{J}}$, in $\mathfrak{S}$, does not annihilate $x_{n+1} \notin [x_1, \cdots, x_n]$, we conclude that the *module annihilator* of $\overline{\mathfrak{J}}$ (the $\mathfrak{D}$-submodule whose elements are annihilated by $\overline{\mathfrak{J}}$) is the submodule $[x_1, \cdots, x_n]$. It yields the following

Theorem 48: *If $\mathfrak{S}$ is an irreducible ring of endomorphisms of $\mathfrak{M}$ and $\mathfrak{D}$ is its commutator, the finite $\mathfrak{D}$-subspaces of $\mathfrak{M}$ are exactly the modules annihilators of the right ideals of $\mathfrak{S}$ which annihilate them.*

11) **On closed rings** — The notion of closed ring could be given in this way: $\mathfrak{R}$ is a closed ring, if there exists a module $\mathfrak{M}$ in whose absolute $\mathfrak{A}$ the ring $\mathfrak{R}$ is faithfully represented and such that, also in $\mathfrak{A}$, $\mathfrak{R}$ and $\overline{\mathfrak{R}}$ are reciprocal commutators. In this sense of closed ring, we can give the following general proposition: $\mathfrak{R}$ is closed, if, and only if, it has the identity.

The greatest interest of the notion of closed ring is given, now, by a more restrictive feature: $\mathfrak{R}$, faithfully represented as an endomorphism ring of some module $\mathfrak{M}$, is closed, if, and only if, $\mathfrak{R}$ and $\overline{\mathfrak{R}}$ are reciprocal commutators.

When $\mathfrak{R} = \mathfrak{D}$ is a division ring of endomorphisms of $\mathfrak{M}$, containing the identity endomorphism, we have shown that $\mathfrak{D}$ is closed. In this §, we will prove a affirmation, made after theorem 42, which respects to a simple noetherian ring of endomorphisms. But, before that, we will return to simple rings with a minimal right ideal and to a question related with the theory of representations of the simple noetherian rings.

If $\mathfrak{S}$ is a simple ring, not zero ring, let us suppose that $\mathfrak{r}$ has a minimal right ideal $\mathfrak{r} \neq (0)$. $\mathfrak{S}$ induces in $\mathfrak{r}$ an endomorphism ring $\mathfrak{S}_1$: $\mathfrak{S} \sim \mathfrak{S}_1$, $x \in \mathfrak{r}$, $x \to xa = xA$, $(a \in \mathfrak{S}, A \in \mathfrak{S}_1)$. As $\mathfrak{S}_1 \simeq \mathfrak{S}/\mathfrak{a}$, where $\mathfrak{a}$ is caracterized by the relation $\mathfrak{r}\mathfrak{a} = (0)$, we have $\mathfrak{a} = (0)$ or $\mathfrak{a} = \mathfrak{S}$. If $\mathfrak{a} = \mathfrak{S}$, we conclude $\mathfrak{r}\mathfrak{S} = (0)$, $\mathfrak{r}^2 = (0)$, $\mathfrak{S}^2 = (0)$, against the hypothesis. Consequently, we have $\mathfrak{a} = (0)$, $\mathfrak{S} = \mathfrak{S}_1$. The simple ring is, then, faithfully represented as irreducible ring of endomorphisms of $\mathfrak{r}$. We have, then, an irreducible ideal ring, where the minimal right ideals are isomorphic, and, by that, $\mathfrak{S}$ equals its anti-radical, (cf. [25]), and can be represented as direct discrete sum $\mathfrak{S} = \Sigma e_\mu \mathfrak{S}$, $(\mu \in M)$, of $\mathfrak{S}$-isomorphic submodules, each one represented by a primitive idempotent. The ring of $\mathfrak{S}$-endomorphisms of $\mathfrak{S}$ is the ring of all row-summable matrices over the division ring $\mathfrak{D}$ of endomorphisms of a simple right ideal. $\mathfrak{D}$ is anti-isomorphic to $e_\alpha \mathfrak{S} e_\alpha$, $(\alpha \in M)$. $\mathfrak{S}$ has also minimal left ideals, which are all isomorphic. And we can give the following

Theorem 49: *If $\mathfrak{S}$ is a simple ring, not zero ring, with a minimal right ideal, the ring of its $\mathfrak{S}$-endomorphisms is isomorphic to a complete ring of row-summable matrices over the division ring of $\mathfrak{S}$-endomorphisms of a minimal right ideal of $\mathfrak{S}$.*

The simple ring $\mathfrak{S}$ of this theorem is not, generally, a closed ring, in any of the two sense of this notion, as it has not, in general, the unity. If $1 \in \mathfrak{S}$, the sum $\Sigma e_\mu \mathfrak{S}$ is a finite one and $\mathfrak{S}$ is a noetherian simple ring, closed in the first of the two senses.

With respect to the second of the senses, one can give the following proposition, connected with the theory of the representations:

Theorem 50: *A noetherian simple ring is closed in the absolute of every of its minimal right ideals.* As we know, it is $\mathfrak{S} = \sum_{i,j} \mathfrak{D}' e_{ij}$, where the e_{ij} constitute a system of matrices units and $\mathfrak{D}'$ is the division ring of those elements of $\mathfrak{S}$ which commute with the matrices units, [(I), pg. 37-39 and 54-58]. We know also that $\mathfrak{D}' \simeq e_{11} \mathfrak{S} e_{11}$ and

$$\mathfrak{S} = \sum_{i,j} \mathfrak{D}' e_{ij} = \sum_i (\sum_j \mathfrak{D}' e_{ij}) = \sum_i \mathfrak{r}_i,$$

where $\mathfrak{r}_i = \sum_j \mathfrak{D}' e_{ij} = e_{ii} \mathfrak{S} = e_i \mathfrak{S}, (e_{ii} = e_i)$, is a simple right ideal. The expression given to $\mathfrak{S}$ shows that $\mathfrak{S}$ is a left $\mathfrak{D}'$-module. Though $\mathfrak{D}'_1$ as it commutes with the e_{ij}, can be placed at right, we will fixe the place of $\mathfrak{D}'$, to give a sense to the application of the product $a'b' \in \mathfrak{D}'$ to the e_{ij}. The convention is that b' acts in first place. Then, $(a'b') e_{ij} = e_{ij}(a'b') \neq e_{ij}(b'a')$, generally. Thus $\mathfrak{r}_1 = \mathfrak{D}' e_{11} + \cdots + \mathfrak{D}' e_{1n}$ is a module which faithfully represents $\mathfrak{S}$ as an irreducible ring of endomorphisms. For $x_1 \in \mathfrak{r}_1$, the correspondence $x_1 \to x_1 A, (A \in \mathfrak{S})$, is an admissible one with respect to $\mathfrak{D}'$: $a' x_1 \to a' x_1 \cdot A = a' \cdot x_1 A$, $(a' \in \mathfrak{D}')$. The elements of $\mathfrak{S}$ induce $\mathfrak{D}'$-endomorphisms in $\mathfrak{r}_1$. We will show that any $\mathfrak{D}'$-endomorphism is represented by an element of $\mathfrak{S}$. If the endomorphism can be given by the correspondence

$$e_{1i} \to e'_{1i} = \sum_k d'_{ik} e_{1k}, \quad (d'_{ik} \in \mathfrak{D}'),$$

taking $s = \sum_{j,k} d'_{jk} e_{jk}$, we have

$$e_{1i} s = \sum_k d'_{ik} e_{1k} = e'_{1i}.$$

Shortly: $\mathfrak{S}$ is faithfully represented in the absolute $\mathfrak{A}_1$ of the endomorphisms of $\mathfrak{r}_1$, as the ring of $\mathfrak{D}'$-endomorphisms. $\mathfrak{D}'$ has in $\mathfrak{A}_1$ an anti-isomorphic image $\mathfrak{D}$, which is a division ring with $\mathfrak{S}$ as its commutator. And from the theory of the irreducibles rings, we know that $\mathfrak{S}$ and $\mathfrak{D}$ are reciprocal commutators.

With the terminology of the theory of the representations, as referred in § 5, the theorem which we have proved means that to the minimal right ideals of a simple ring $\mathfrak{S}$, completely reducible and with identity, belong faithful reciprocal and irreducible representations of $\mathfrak{S}$. The representation ring is the division ring $\mathfrak{D}$ of the right $\mathfrak{S}$-endomorphisms of each of the minimal right ideals, which behaves as double-modules for which $\bar{\mathfrak{D}}=\mathfrak{S}$, $\bar{\mathfrak{D}}=\bar{\mathfrak{S}}=\mathfrak{D}$. Clearly, as it happens in the theory of reciprocal representations by matrices, $\mathfrak{D}$ and $\mathfrak{S}$, are right operators of the representation module $\mathfrak{r}_1$, [(1), Cap. VIII].

It follows the proposition referred in the beginning of this § and after thorem 42. It reports to an important case of closed ring and constitute a generalization of the theory of division rings of endomorphisms. We have:

Theorem 51: *If $\mathfrak{S}$ is a noetherian simple ring of endomorphisms, containing the identity endomorphism, $\mathfrak{S}$ and $\bar{\mathfrak{S}}$ are reciprocal commutators in the absolute $\mathfrak{A}$* (or: *$\mathfrak{S}$ is closed in $\mathfrak{A}$*). According the considerations which preceded theorem 36, let us write $\mathfrak{M}=\Sigma\,\mathfrak{m}_\mu$, $(\mu \in M)$, or, more precisely, $\mathfrak{M}=\Sigma\, x_\mu \mathfrak{r}_\mu$, where the $x_\mu \in \mathfrak{M}$ and the $\mathfrak{r}_\mu$ are simple right ideals of a decomposition of $\mathfrak{S}$. Each $\mathfrak{m}_\mu$ is an $\mathfrak{S}$-module, $\mathfrak{S}$-closed, as it follows of theorem 50, by corollary 10 of theorem 31, $\mathfrak{M}=\Sigma\,\mathfrak{m}_\mu$ is an $\mathfrak{S}$-module, $\mathfrak{S}$-closed.

As in the case of division rings, we can make a direct verification of the theorem. The isomorphisms $\Delta_{\mu\nu}$, introduced in § 8, in the theory of direct discrete sums of isomorphic modules, in which $x_\mu \mathfrak{r}_\mu \simeq x_\nu \mathfrak{r}_\nu$, can be defined in this way: let us consider an isomorphism $\mathfrak{r}_\mu \simeq \mathfrak{r}_\nu$, in which $e_\mu \to \rho_\nu$, and let us take $x_\mu e_\mu \to x_\nu \rho_\nu$, $x_\mu e_\mu s \to x_\nu \rho_\nu s$, $(s \in \mathfrak{S})$. If $\bar{\mathfrak{S}}_\mu$ and $\bar{\mathfrak{T}}_\mu$ are the rings of $\mathfrak{S}$-endomorphisms of $\mathfrak{r}_\mu$ and $x_\mu \mathfrak{r}_\mu$, respectively, we know [§ 8, theorem 31] that $\bar{\mathfrak{S}}$ is isomorphic to the ring of the $\bar{\mathfrak{S}}_\mu$-endomorphisms of $\mathfrak{r}_\mu$, which, by theorem 50, is isomorphic to $\mathfrak{S}$. Then $\bar{\mathfrak{S}}$ and $\mathfrak{S}$ are isomorphic. A more detailed analysis shows that to every $\Theta \in \bar{\mathfrak{S}}$ corresponds an $s \in \mathfrak{S}$ such that $\Theta = s$, and the theorem is proved. Let us take $\Theta \in \bar{\mathfrak{S}}$ and let $E_\mu \in \bar{\mathfrak{S}}$ be the $\mathfrak{S}$-endomorphism which applies $\mathfrak{M}$ over $x_\mu \mathfrak{r}_\mu$. Θ defines an $E_\mu \bar{\mathfrak{S}} E_\mu$-endomorphism in $x_\mu \mathfrak{r}_\mu$, to which corresponds a $\bar{\mathfrak{S}}_\mu$-endomorphism of $\mathfrak{r}_\mu$. This one, according theorem 50, is represented by a right multiplication by an $s \in \mathfrak{S}$: $r_\mu \to r_\mu s$. The corresponding $\bar{\mathfrak{T}}_\mu$-endomorphism is then $x_\mu r_\mu \to x_\mu r_\mu s = (x_\mu r_\mu)\Theta$, [cfr. theorem 31]. We will now prove that $x_\nu r_\nu \to (x_\nu r_\nu)\Theta = x_\nu r_\nu s$, for every ν. Let then $e_\mu \to \rho_\mu$, $e_\mu t \to \rho_\nu t = r_\nu$. We have

$$x_\mu e_\mu t \to (x_\mu e_\mu t)\Delta_{\mu\nu} = (x_\mu e_\mu t) E_\mu \Delta_{\mu\nu} =$$
$$= (x_\mu e_\mu t) E_{\mu\nu} = x_\nu \rho_\nu t = x_\nu r_\nu;$$

$$x_\nu r_\nu \to (x_\nu r_\nu)\Theta = (x_\nu \rho_\nu t)\Theta = (x_\mu e_\mu t) E_{\mu\nu}\Theta =$$
$$= (x_\mu e_\mu t)\Theta E_{\mu\nu} = (x_\mu e_\mu ts) E_{\mu\nu} = (x_\mu e_\mu t) E_{\mu\nu} s = x_\nu r_\nu s.$$

The theorem now proved is included in the following general proposition, which is nothing more that corollary 10 of § 8:

Theorem 52: *If $\mathfrak{B}$ is an endomorphism ring of a module $\mathfrak{M}$, containing the identity endomorphism, $\mathfrak{B}$ and $\bar{\mathfrak{B}}$ will be reciprocal commutators in the absolute $\mathfrak{A}$ of $\mathfrak{M}$, if we can write $\mathfrak{M}=\Sigma\,\mathfrak{m}_\mu$ as a direct discrete sum of $\mathfrak{B}$-submodules, $\mathfrak{B}$-isomorphic, in which one $\mathfrak{B}$ is closed.*

Let now $\mathfrak{S}$ be a simple ring, with the field $\mathfrak{Z}=\mathfrak{P}$ as its center. If $\mathfrak{A}_{\mathfrak{P}}$ is the ring of $\mathfrak{P}$-endomorphisms, we have $\mathfrak{A}\supseteq\mathfrak{A}_{\mathfrak{P}}\supseteq\mathfrak{T}$. The subrings $\mathfrak{S}_r$ and $\mathfrak{S}_l$ are reciprocal commutators in $\mathfrak{A}$, $\mathfrak{A}_{\mathfrak{P}}$ and $\mathfrak{T}$. In $\mathfrak{A}$ and $\mathfrak{A}_{\mathfrak{P}}$, $\mathfrak{P}$ and $\mathfrak{A}_{\mathfrak{P}}$ are reciprocal commutators; and, in $\mathfrak{T}$, $\mathfrak{P}$ and $\mathfrak{T}$ are also reciprocal commutators. If $\mathfrak{P}$ is a prime field, the results on irreducible rings show that:

THEOREM 53: *In a simple ring* $\mathfrak{S}$, *whose center is a prime field* $\mathfrak{P}$, *all endomorphisms are* $\mathfrak{P}$-*endomorphisms.*

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